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Some Considerations Concerning the Teaching of Negative Numbers

Algumas Considerações sobre o Ensino de Números Negativos

Algunas Consideraciones sobre la Enseñanza de los Números Negativos

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ABSTRACT

The teaching of negative numbers is dealt with herein from the point of view of expanding the number classes from those containing only positive numbers to those that also include the negatives (and zero). The expansion involves generalizing the concept of "number" and making modifications in the definitions of the arithmetical operations. Each of these actions generates cognitive obstacles, both historically and pedagogically. In contrast to the historical situation, the concept of negative number is not problematic for the contemporary student because he/she encounters them with certain frequency in his/her daily life. The addition of negative numbers, especially in the case in which it becomes equivalent to subtraction in the natural numbers with the subtrahend greater than the minuend, is subject to a didactical obstacle, which can be overcome by using the number line as a visual/conceptual support. Also, the multiplication of two negative numbers, resulting in a positive number, constitutes a genuine epistemological obstacle due to the fact that the usual metaphors used to explain the negative numbers do not clarify this operation. The article thus proposes to consider double negatives as inverses of inverses and delineates a sequence of activities based on this metaphor to overcome the mentioned epistemological obstacle.
Keywords: Teaching of arithmetic; Cognitive obstacles; Negative numbers; Rule of signs; Inverse of the inverse.

RESUMO

O ensino dos números negativos é abordado no presente artigo do ponto de vista da expansão das classes numéricas de elas que só contêm números positivos para elas que também incluem os negativos (e zero). A expansão envolve a generalização do conceito de "número" e a modificação das definições das operações aritméticas. Cada ação desta gera obstáculos cognitivos, tanto historicamente, quanto pedagogicamente. Em contraste à situação histórica, o conceito de número negativo não é problemático para o estudante contemporâneo porque ele/ela o encontra na sua vida quotidiana. A soma de números negativos, especialmente quando toma a forma equivalente à subtração de números naturais com o subtraendo maior do que o minuendo, é suscetível a um obstáculo didático, que pode ser superado por usar a linha numérica como um apoio visual/conceitual. Ainda mais, a multiplicação de dois números negativos, resultando num número positivo, constitui um obstáculo epistemológico genuíno devido ao fato de que as metáforas geralmente usadas para explicar os números negativos não esclarecem essa operação. O presente artigo propõe considerar negativos duplos como inversas de inversas e delineia uma sequência de atividades baseadas nessa metáfora para superar o referido obstáculo epistemológico.
Palavras-chave: Ensino de aritmética; Obstáculos cognitivos; Números negativos; Regra dos sinais; Inversa da inversa.

RESUMEN

La enseñanza de los números negativos se aborda en este artículo desde el punto de vista de la expansión de clases numéricas desde las que solo contienen números positivos a las que también incluyen negativos (y cero). La expansión implica generalizar el concepto de "número" y modificar las definiciones de las operaciones aritméticas. Cada una de estas acciones genera obstáculos cognitivos, tanto histórica como pedagógicamente. A diferencia de la situación histórica, el concepto de número negativo no es problemático para el estudiante contemporáneo porque lo encuentra en su vida cotidiana. La suma de números negativos, especialmente cuando toma la forma equivalente a la resta de números naturales con el sustraendo mayor que el minuendo, es susceptible a un obstáculo didáctico, que puede superarse usando la recta numérica como ayuda visual/conceptual. Además, la multiplicación de dos números negativos, dando como resultado un número positivo, constituye un verdadero obstáculo epistemológico debido a que las metáforas generalmente utilizadas para explicar los números negativos no aclaran esta operación. Este artículo propone considerar las dobles negativas como inversos de inversos y esboza una secuencia de actividades a partir de esta metáfora para superar el mencionado obstáculo epistemológico.
Palabras clave: Enseñanza de la aritmética; Obstáculos cognitivos; Números negativos; Regla de los signos; Inversa de la inversa.

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INTRODUCTION

One of the most endearing – although some would say "notorious" – aspects of mathematics is its propensity for abstraction, which can indeed occur in at least two ways. First of all, a concept may be widened so that its sense becomes more inclusive and/or its reference embraces individuals not contemplated by the older usage. Secondly, certain procedures or techniques may be extended, suitably tweaked when necessary, to new domains. Most of the time, perhaps, the two go hand-in-hand, as in the case of negative numbers. Since the resulting abstraction generally acquaints us with mathematical landscapes with which we are unfamiliar, it ofttimes presents us with various cognitive obstacles. Such was (is!) the case with negative numbers: their introduction into mathematics occasioned two principal obstacles, one for each type of generalization. The former has been largely overcome, but the latter has remained to plague our elementary school students. Herein, we will take a closer look at each of these obstacles and suggest, for the latter, some activities designed to overcome the problem. First, however, it will be useful to recapitulate the development of the number classes and to make some remarks about cognitive obstacles.

BRIEF RECAPITULATION OF NUMBER CLASSES

Pedagogically, the number classes are arranged, at least implicitly, as a series of supersets $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$, in which each superset is considered as some kind of closure of the preceding subset, as is summarized in Table 1[2]:

Table 1 – Number classes as closures

Superset	Closure of	With Regard to	Example
$\mathbb{Z}$	$\mathbb{N}$	subtraction	$4 - 7$
$\mathbb{Q}$	$\mathbb{Z}$	division	$\frac{13}{5}$
$\mathbb{R}$	$\mathbb{Q}$	limits	$\sqrt{11}$
$\mathbb{C}$	$\mathbb{Q}$	root-taking	$4i$

Source: Elaborated by the autor.

This arrangement is, in a way, rather nice, since it presents each new superset as resulting from a perceived need in the mathematical community and sets out the various number classes in a logical progression. I rather think, however, that it is more appealing to those who have already constructed a more or less sophisticated understanding about the mathematics underlying the relations among these classes. Most learners would probably better appreciate a more historical account of the development of these classes since said account brings in more lively considerations and problems with which the student can grapple as if he/she were at the cutting edge of the debate in the historical moment. This, in its turn, promotes more dynamic instructional methodologies and fosters the development of the student as an active agent in the construction of his/her mathematical knowledge. (See, for, example, MENDES; FOSSA; VALDÉZ, 2006).

Historically, the first numbers to appear were the natural numbers, , or, as they are often called, the "counting numbers":

2 There are, of course, other kinds of numbers – the transfinite numbers, for example –, but they involve other kinds of considerations and are not really relevant to our concerns herein.

$$\mathbb{N} = \{1, 2, 3, 4, \ldots\}.$$

Some have argued that zero also belongs to this set. Lumpkin (2005), for example, claims that there is a zero-line in ancient Egyptian use of level lines in many of their works of art. That may well be, but it does not mean that zero was considered a *number*. Indeed, there was no need for a zero counting number, because, if there's nothing there, there's nothing to count.

But even the "natural" numbers are not natural in the sense that they are somehow given in nature. They had to be built up and different cultures built them up in different ways, incorporating, for example, different bases and/or subbases in the formation of their number words and number symbols. They were not always consistent in doing so. Old French was consistent in using base ten words for 70, 80 and 90, *viz*., *septante*, *octante* (or *huitante*) and *nonante*. Modern French, probably due to borrowings, uses an additive principle for 70 (*soixante-dix*) and base twenty words for 80 (*quatre-vingt*) and 90 (*quatre-vingt-dix*)[3]. Many cultures use classifiers for different kinds, or even different shapes, of objects and may also use different number words for different objects. Irish Gaelic, in this regard, is relatively well behaved in that it has but three systems: one for the abstract sequence of numbers, one for counting things and one for counting people. The number 2 is *a dó*, whereas for 2 things and for two people *dha* and *beirt* are used respectively. These words also affect the grammar in different ways. The singular is used after object numbers, but the plural after people numbers: *dha chat* (instead of *chait*) for "two cats", but *beirt chailíní* (instead of *chailín*) for "two girls". Strangely enough the numbers from 1 to 6 cause lenition (aspiration) *chat* and instead of *cat*, whereas 7 to 10 cause eclipsis, so that "seven cats" is *seacht gcat*. Other cultures have a much more complicated system of number classifiers. Numerals also vary from culture to culture, as is exemplified by Indo-Arabic and Roman numerals. For more details on the great variety of number words and number systems, see Fossa (2010).

After the somewhat convoluted development of the natural numbers, one would expect, following the pedagogical (logical) model, the negative numbers to come next. In point of fact, however, the rationals were the next to appear. In contrast to the counting numbers, these may be thought of as measuring numbers because, instead of being used to count discrete quantities, they are used to measure continuous magnitudes. The ancient Greeks seem not to have thought of the rationals as numbers but as the ratio of two numbers. Nevertheless, they soon discovered the reals – the negatives were still awaiting their turn! – in the form of irrational surds. Now, for a discipline that prides itself on using the strictest logic, the name "irrational" must sit a bit uneasily. It actually comes from the Greek word *lógos*, which means, primordially, "word", but which had a veritable plethora of uses. Thus, it could also mean, among other things, "speech", "rational speech", the "plot" of a story, an "analogy", a "reason" for something, a "resolution" or "intention", a "rule", "principle" or "law". In mathematics, it also meant a "ratio". Thus, the rationals were just the set of ratios of two numbers. Surds, in contrast, were magnitudes that were not ratios, that is they were *a-lógos*, the *a* being the "*a-privative*". Our term (via the Latin) is, thus, just *in-rational* > *ir-rational*, that which is not a ratio (the Latin *in* corresponding to the Greek "*a-privative*"). The Greeks, as well

3 The *dix* additive is the subbase.

as other ancient peoples knew of at least one whole different kind of irrational, the constant π, although this was shown to be irrational and transcendent only much later.

Negative numbers were used in computation in ancient China, though just how far back they go is still controversial. Diophantus (200-284) used them in the same way. In neither case do they seem to have been used as stand-alone numbers. As time went on, however, they became more and more useful in both commercial and scientific applications of mathematics. Still, they were only fully incorporated into mathematics in the 19th Century. I will return to these topics in more detail in the next section but one. Of course, once the negatives were contemplated, their square roots were sure to follow in relatively short order.

OBSTACLES IN EDUCATION

The concept of "epistemological obstacle" (*obstacle épistémologique*) was introduced in Gaston Bachelard (1938). Along with its sister concept, "epistemological break" (*rupture épistémologique*), it was used in Bachelard's analysis of the history of science. In this regard, it influenced various philosophers of science, perhaps most notably Thomas Kuhn (1962) and the concept of "paradigm shifts" in science. According to Bachelard the very structure of what Kuhn would call "normal science" could become an obstacle to scientific advancement by inhibiting the comprehension and eventual acceptance of new ideas.

Bachelard's analysis also received a hearty welcome in another domain: that of education, in general, and Mathematics Education, in particular. It also seemed to be in sync with psychological theory, for the epistemological break, or paradigm shift, could be conceptualized as a large-scale Piagetian accommodation – the restructuring of conceptual networks. In the field of education, however, "epistemological obstacle" soon took on a variety of different acceptations, which were eventually sorted out as different kinds of obstacles. I mention a few, by way of example:

> **epistemological obstacles** – inhibitions due to the structure of the student's cognitive base;
> **didactical obstacles** – inhibitions due to the teacher's presentation;
> **qualificational obstacles** – inhibitions due to inadequacies in the student's cognitive base;
> **attitudinal obstacles** – inhibitions due to the student's emotional response towards the material being taught ("fear of mathematics"!) or the teaching situation;
> **physiological obstacles** – inhibitions due to medical deficiencies, including, and in some cases, especially, those due to inadequate nutrition;
> **economic obstacles** – inhibitions due to lack of means for the obtention of basic materials to be used in the classroom;
> **discriminatory obstacles** – inhibitions due to bigotry of any kind;
> **material obstacles** – inhibitions due to inadequacies in the physical plant of the school;*etc*.

I tend to refer to any or all of these as **cognitive obstacles**. In the following two sections, I will discuss, sequentially, some of the cognitive obstacles occasioned by the two types of abstraction in the teaching/learning of negative numbers in the grammar school setting. The reader is invited to compare and contrast the (more detailed) account of Schubring (2018).

THE CONCEPT OF NEGATIVE NUMBERS

As already mentioned, the negative numbers were used implicitly in Antiquity and their usefulness in certain applied contexts became more and more apparent as time went on. Nevertheless, they were not readily incorporated into mathematical theory. Indeed, according to the ancients' account of number, the negatives did not fit in easily, for number was a collection of units and the unit (1), of course, was unique (see, for example, FOSSA e ANJOS, 2007). This way of taking number had behind it all the incomparable authority of Euclid (1956, VII, Def. 2), who, in Heath's translation, stipulated that

A number is a multitude composed of units.

At first, the rationals and surds were assimilated by regarding the first as ratios and the second as geometric magnitudes. Eventually, however, both were accommodated, for all practical purposes, in a more generalized number concept. Embarrassing questions could be avoided by giving only a vague description of "number" in general – for example, Barlow (1814) states that

> NUMBER, in its most extended signification, has a reference to every abstract quantity that can be made the subject of arithmetical computation;

– and leaving strict operational definitions for the different kinds of number.

One way of outflanking the problem regarding negative numbers was to subsume the negativity into the units of measurement used in any specific problem. This would be akin to treating the rationals as submultiples. In doing so, however, one would hopelessly multiply non-mathematical considerations in purely mathematical contexts and it was, thus, much more economical, for mathematical theory, to have a unified number system than to contemplate a fractured, multipartite system of weights and measures. Analogously, pressure for the acceptance of negative numbers as legitimate numbers came from mathematical theory, this time from the theory of equations. The naysayers, however, resisted with redoubled enthusiasm because, they claimed, the very idea of negative numbers was logically incoherent. The controversy that ensued, ably chronicled in, for example, Pycior (1997) lasted centuries.

The incoherence of the negative numbers derived from their being conceived of as quantities less than nothing. Thus, Barlow (1814), in his entry for "negative", states that

> It is vain to attempt to define what can have no possible existence; a quantity less than nothing is totally incomprehensible; and to illustrate it, by reference to a debtor and creditor account, to say the least of it, is highly derogatory to this most extensive and comprehensive science.

We should note that this is indeed a very powerful argument. There is no thing that is less than nothing because, if there were, it wouldn't be nothing after all. Some mathematicians nonetheless still wanted to embrace negative numbers despite the apparent logical impasse and the reason they wanted to do so was not one of mere facility of calculation. Mathematicians had become aware, by the early 17th century, that there was a connection between the degree of a polynomial equation and the number of roots it has. This was such a striking and important result that it was termed The Fundamental Theorem of Algebra. Unfortunately, for it to be true, one needed negative and complex numbers to serve as roots. Some were more enamored with the theorem than repulsed by the logical difficulty, while others were more respectful of the logic than charmed by the theorem.

The controversy was effectively put to rest by George Peacock's axiomatization of the integers in 1830 (see PEACOCK, 2004), for it presented algebra as a science of uninterpreted symbols so that number need not be considered as an ontological quantity. This undercut the basic premise of the naysayers because, by freeing the concept of number from its associations with physical quantities and considering numbers as abstract entities in an abstract mathematical system, it was no longer allowable to *identify* zero with nothing. Thus, the way was paved for the acceptance of negative and imaginary numbers.

Curiously, the contemporary student has but little difficulty in accepting the concept of negative numbers. This is probably due to two reasons. First, negative numbers are prevalent in most student's circumambient culture[4], especially in regard to temperature, but also in other contexts. Second, at the age the student is introduced to negative numbers, he/she usually does not have the philosophical sophistication necessary to confuse "number less than zero" with "quantity less than nothing" and therefore does not engender unnecessary muddles for him/herself.

This is a truly informative observation because one of the ways that the History of Mathematics can be used in Mathematics Education is that of observing where cognitive obstacles occurred in the past in order to detect – and hopefully overcome – them in the present-day classroom. *Sometimes*, however, historical obstacles just do not reappear in the present. It also gives us pause when we try, as indicated above, to imaginatively put the student "at the cutting edge of the debate in the historical moment" because his/her cognitive base may contain relevant items not present to the original actors. This is not, of course, a reason for not using history as a pedagogical tool, but for using it conscientiously and creatively.

COMPUTATION WITH NEGATIVE NUMBERS

Before the acceptance of negative numbers, we do not have the pedagogical arrangement of the number classes as given above. Rather, we have the following abridgement thereof:

$$\mathbb{N} \subset \mathbb{Q}^{+} \subset \mathbb{R}^{s}$$

where is an amorphous subset of the reals, limited, for the most part, to the surds and π. When the negatives are included, we are all of a sudden confronted with the symmetric values of all these numbers, as well as, of course, zero. The door is also opened to the imaginaries, but they do not really concern us here, except incidentally. The influx of all these new numbers raises the question of how we are to perform the arithmetical operations with them. A basic restraint on solving this problem is that of insuring that the old operations be

4 It would, however, be interesting to make a comparative study of students in large or mid-sized cities and those in rural areas of tropical countries.

retained in each of the aforementioned subsets of the new supersets. That is, for example, the addition of two positive integers of $\mathbb{Z}$(as well as of $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$) should give the same result as the addition of these same two integers in $\mathbb{N}$.

Clearly, it will not be necessary to give the details of how this is done here. Supposing it done in the usual manner, I will now consider some of the most important cognitive obstacles that the new operations may give rise to for the student. This will be done under two headings, that of addition (including subtraction) and that of multiplication (including division), as each of them gives rise to different considerations.

Addition

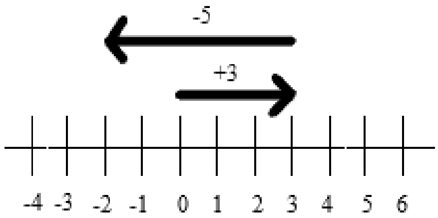
The first thing to observe in the new number domains is that the signs $+$ and $-$ now take on a double meaning. Erstwhile symbols just for indicating operations, they now also indicate whether a number be positive or negative. Further, except for the initial presentation of the system, the $+$ sign as a sign of positive numbers drops out so that operationally there is no difference between $\mathbf{7}+(-\mathbf{3})$ and $\mathbf{7}-\mathbf{3}$. Actually, the system is quite elegant and is easily mastered by most students in short order. Parentheses are very helpful; expressions such as $\mathbf{7}+-\mathbf{3}$ should be avoided. This only leaves stand-alone negatives $(-\mathbf{3})$ and double negatives $(- \ -\mathbf{3},$ or better, $-(-\mathbf{3}))$ as potentially problematic. But we have seen that the stand-alone negatives are already familiar to most students and, given that $-(-\mathbf{3}) = (-\mathbf{1})\times(-\mathbf{3})$, they are more properly treated under multiplication.

Subtraction, in the context of the natural numbers, may give rise to a didactical obstacle, when the teacher tells the student that "you can't take the more from the less". Of course, you can't in $\mathbb{N}$, $\mathbb{Q}^+$, *etc*. In moving to more generalized contexts like $\mathbb{Z}$, $\mathbb{Q}$, *etc*., this restriction is removed, but if the student has internalized the impossibility of taking the greater from the smaller, he/she may have problems in adjusting to the generalized operations. Hence, it may be best to avoid such instructions. Indeed, there also may be other reasons to avoid the greater from the lesser prohibition. Consider the following word problem:

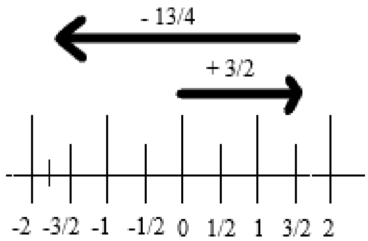
> Matilda got three marbles from Clotilda. If Clotilda had seven marbles, how many does she have now?

This is a difficult problem because the word "got" may indicate, to the student, addition instead of subtraction. Once the correct operation is elected, however, many students take the order of the operands to be that which is given in the problem and write $\mathbf{3}-\mathbf{7}$. The teacher is naturally tempted to "explain" that you can't take more from less. This, however, is not a real explanation, but rather a handy trick which indeed helps the student obtain the correct numerical expression, but which does so at the cost of him/her not coming to grips with the semantical content of the problem. Naturally, reliance on such tricks diminishes the student's prowess as a problem solver and tends to promote the student's impression of mathematics as arbitrary contrivances, instead of reasoned procedures for understanding the world about us.

Perhaps one of the best cognitive supports for the addition of negative numbers is still that of moving along the number line. The sum $\mathbf{3}-\mathbf{5} = -\mathbf{2}$, for example, can be pictured graphically as



There is, of course, no reason to limit the number line to the addition of integers. Thus, the sum can be pictured as



The visual representation gives the student an intuitive understanding of what it means to add negative numbers. Once that has been attained, however, he/she should be weaned from this support.

Multiplication

Usually hardly anyone has any problem with three-fourths of the "rule of signs". Schematically, $+ \times + = +$ (which is just the old multiplication of positive numbers), and $+ \times - = - \times + = -$ are unproblematic. The remaining clause, $- \times - = +$, however, induces extreme befuddlement. Historically, it was also troublesome and, indeed, this is an excellent example of an historical cognitive obstacle that reappears in the contemporary classroom. As is often the case, in expanding the concept of number to include the negatives we had to deal with new entities with which we had not built up any intuitive experience. Moreover, the usual metaphors employed in the discussion of the negatives did not clarify the situation: if, for example, -2 represents a debt of 2 dollars and -3 one of 3 dollars, what could it possibly mean to multiply these two debts by each other? And, indeed, how could one make a profit by doing so!! Although $-3^{o}\mathrm{C}$ makes perfect sense, what could $- -3^{o}\mathrm{C}$ mean? It would, of course, just mean $3^{o}\ \mathrm{C}$ *after* the rule "minus times minus is plus" has been internalized, but it is this very rule that the metaphor was supposed to explain!

One could try to explain multiplication by iterated addition and use the number line to justify the proposed rule of signs, but, as it turns out, doing so is rather tricky. It works better for – as a symbol for an operation, but not so well for – as signalizing a kind of number.

At this point we should observe that we are not trying to prove to the student, mathematically, that $- -a = a$. This is easily done from ring theory (or even group theory), but that is much too abstract to be helpful to our students. Rather, the question is: why did we set up our system in the way we did so that the rule of signs was included in it? Recall that, in expanding our number system to include the new elements (negative numbers), we had to generalize the arithmetical operations subject to the constraint that the old positive number arithmetic would still work in the new system. This constraint is satisfied by stipulating that we make $- \times - = +$, but is not satisfied by making $- \times - = -$. Take the distribution of multiplication over addition, for example. If minus times minus is to be minus, we obtain

$$-6 = -3(2) = -3(4-2) = -3(4)+(-3)(-2) = -12-6 = -18$$

Such a result is abhorrent (in the usual arithmetic) because –6 is assuredly not –18. Upon taking minus times minus to be plus, in contrast, everything works out as it should:

$$-6 = -3(2) = -3(4-2) = -3(4)+(-3)(-2) = -12+6 = -6$$

Thus, the answer to our question about why we set up our system to include the rule of signs is just that it is what is needed to make everything work out the way we want it to.

Whereas the answer given in the last paragraph is completely correct and compelling to those who have experience in using arithmetic with negative numbers and/or with the history of mathematics, it is not likely to advance the beginner's intuitive understanding of negative number arithmetic because it demands too much sophistication on the part of the learner. Again, it probably would reinforce his/her conception of mathematics, mentioned above, as arbitrary contrivances. Ironically, a more satisfying approach can be had if we take a more abstract viewpoint of the negatives.

From a formal point of view, the negative numbers are the (additive) inverses of the positive numbers. Looked at in this way, since the inverse of the inverse, as is generally recognized, brings us back to our starting point, it would be quite natural to accept $- -a = a$, once the negative sign be associated with the concept of inverse. The structure of double negatives (as the very term "negative number" indicates) is formally equivalent to that of double negations in logic. Both can be seen as undoing something that was already done: a double negation returns an affirmation, just as a double negative returns a positive number. This, in itself, may be put to good use, depending on the learner's linguistic intuitions. It would probably not be very effective, however, if the learner conceived of, for example, "I don't got no money" as an emphatic way of proclaiming his/her poverty.

In any case, there are many everyday situations in which we are familiar with the inverse of the inverse, though we don't usually think of them in these terms. Take, for instance, the mundane action of turning a shirt inside out. If we repeat the action, the shirt gets right side out again. It only remains, therefore, to turn these experiences to good use by bringing their inverse structure to conscious awareness and interpreting negative number arithmetic in terms of our intuitive knowledge of these experiences. In the next section, I will present a few simple activities that may be used to this end.

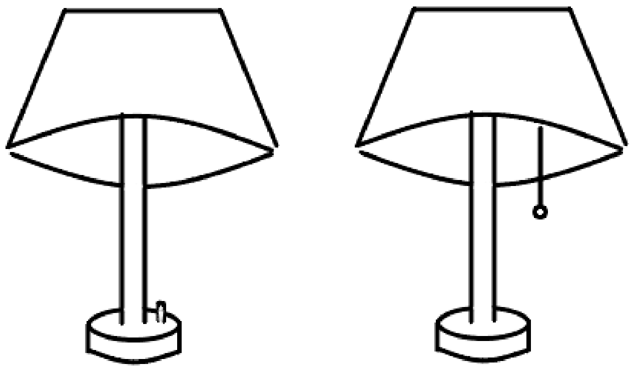
ACTIVITIES FOR THE INVERSE OF THE INVERSE

The following set of activities propose to familiarize the student with the way in which double negatives "work" as inverses by building up a more reflexive awareness of what is involved in the everyday experience of turning on and off electric/electronic devices such as lamps or cell phones and then transferring the insights gained to the mathematical manipulation of negative numbers.

Activity 0

This is a preliminary or warm-up activity, designed to reacquaint the student with the kind of appliance to be used in the following activities, which are all card games. Should lamps be used, for example, they should be lamps with push button or chain pull switches (see Figure 1). Lamps with on/off position or toggle switches should not be used because they would allow the student to determine whether the lamp is on or off by the position of the switch and, therefore, turning the lamp on would be a different operation from turning it off. This would compromise the logic behind the activities.

Figure 1. Lamps



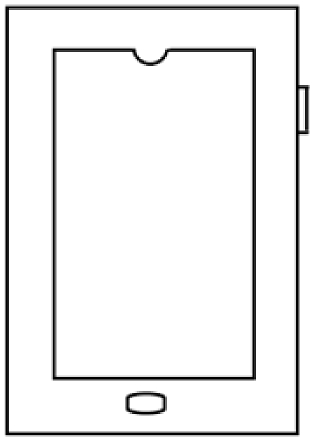
Source: Elaborated by the author

Should cell phones be used, this problem would be obviated because the on/off button of the typical cell phone is of the push button type (see Figure 2). Cell phones have the advantage of being very familiar to most students, but many teachers find them disruptful to the order of the classroom and would rather not use them.

Whatever appliance be chosen, the present activity consists in simply turning it on and off several times in a playful manner. The teacher should point out that the same action turns the appliance from off to on and from on to off and contrast that to other kinds of switches. One need not "make a big deal" out of this, but the student should be made aware of it to help him/her in his/her later reflections.

In the activities that follow, I have elected to use the lamp.

Figure 2. Cell phone



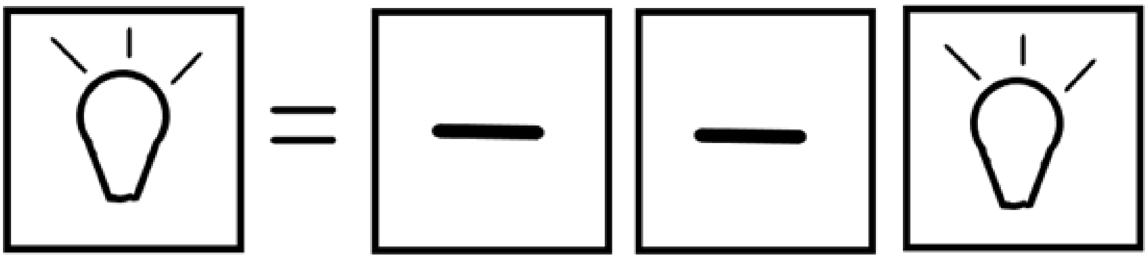
Source: Elaborated by the author

Activity 1

The material involved in this activity is simply a deck of cards, the face side of which shows a lamp, either on or off (these will be referred to as "lamp cards"), or an operation, either + or – (these will be referred to as "operation cards"). The lamp on the lamp cards can be reduced to a lightbulb with or without appropriate "rays of light" showing it to be on or off (see Figure 3).

The cards are to be split into two decks: deck A, containing only lamp cards, and deck B, containing both lamp cards and operation cards. Each player gets the same number of cards from deck B. Play starts when the top card from deck A is turned over. The object of the game is to get rid of all one's cards. This is done by discarding from one's hand 1, 2 or 3 cards which are equivalent to the upturned card from deck A. Figure 3 shows an example.

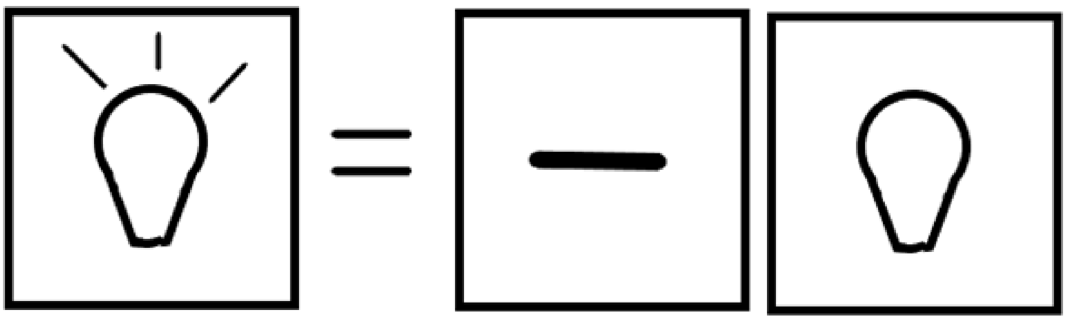
Figure 3. Equivalent cards



Source: Elaborated by the author

From the very nature of the game, it should be evident that the player will make an equivalence by putting down one and only one lamp card and either no, 1 or 2 operation cards. It will also be useful, as well as more fun, to lay the cards down from right to left, using the following terminology: lamp on, lamp off, leave it be[5] (for $+$) and push the button for $-$. Although "leave it be" is just doing nothing to the lamp, in the context of the game it is a genuine operation because it results from a conscious decision and is accompanied by the action of laying down a $+$ card. In laying the cards down from right to left, the player mimics the temporal sequence of actions that would be involved in working with a real lamp and this helps to strengthen his/her intuitions about the use of the operations. Thus, as he/she lays down the cards from right to left in the example given in Figure 3, the player would say "lamp on, push the button, push the button". An alternative equivalence is given in Figure 4. In making this play, the player would say "lamp off, push the button".

Figure 4. Equivalent cards

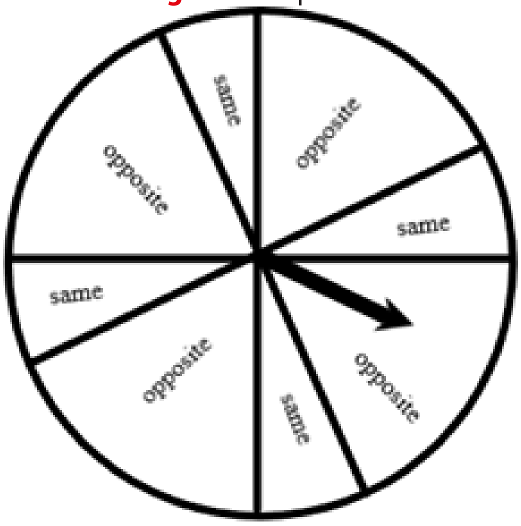


Source: Elaborated by the author

5 Or "let it be" for Beatles' fans.

If a player cannot make a play from the cards he/she has in his/her hand, he/she takes a card from deck B until it becomes possible to do so. The game continues with the second player, who turns over a new card from deck A and proceeds in a similar manner. The winner is, as already indicated, the first player to get rid of all his/her cards, although one could also contemplate ties if two or more players get rid of their cards in the same round.

Figure 5. A spinner



Source: Elaborated by the author

Activity 2

This is the same as Activity 1 with the addition of a spinner. In order to arrange it so that there is a greater chance of getting "opposite", these slices can be made wider (see Figure 5) or a "lose turn" slice could be substituted for a "same" slice.

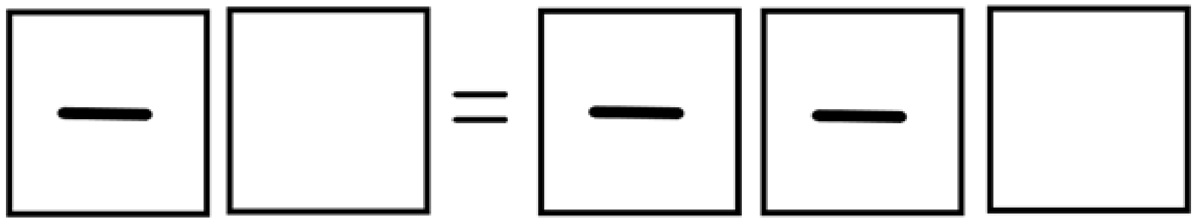
Play begins, as before, by turning over the top card of deck A. The player then spins. If the needle lands on a "same" slice, he/she proceeds as in Activity 1. If it lands on an "opposite" slice, he/she must, in order to discard, build a configuration equivalent to the opposite state from that shown on the overturned card.

The inclusion of a new element of luck with the spinner makes the game more dynamic and, thus, more fun. Moreover, from the pedagogical point of view, it redoubles the player's awareness of the operations to be chosen and, thereby, fortifies even more his/her intuitions about these operations.

Activity 3

This activity uses the cards without the spinner. The teacher, his/her assistant or one of the players is the "dealer" and he/she lays down a proposed equivalence with one or more of the cards face down. The players have to lay down a card, or cards, from their hand that will make a true equivalence. Only those who make a true equivalence are allowed to discard the card/cards played. When the dealer is the teacher or his/her assistant, blank cards may be used. However, when the dealer is one of the students, he/she turns over the card, or cards, he/she has put face down; if they do not make a true equivalence, he/she must pick up all the cards he/she had laid down.

Figure 6. An open equivalence



Source: Elaborated by the author

Figure 6 shows an open equivalence with two blank cards. To make a true equivalence, one needs lay down lamp cards of opposite states: either lamp on for card 2 and lamp off for card 5 or lamp off for card 2 and lamp on for card 5. Here two lamp cards were left blank, but it is permitted to leave blank any combination of lamp cards and operation cards that strikes the player's fancy.

As should be clear, this activity is a step toward greater abstraction. A further step in this direction will be taken in the next activity.

Activity 4

By now the student should be ready to ditch the lamps and contemplate abstract strings of $+$ and $-$. Thus, in the present activity, only operation cards will be used. A template like that of Figure 6, but having all the places blank will be used to start the game. Each player, in turn, lays down an operation card on a blank space, subject to the constraint that the result be a true equivalence. Since the result will only be determined by the final card and given that there are 5 blank spaces, there should be 2, 3 or 4 players so that everybody gets a chance to lay down a final card in the equivalence. If a player cannot make a true equivalence, he/she must draw from the deck until a true equivalence can be made. As in the other activities, the first player to get rid of all of his/her cards is the winner.

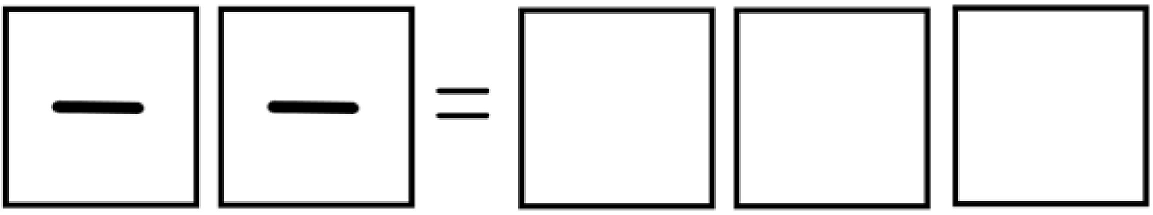
In this activity, the lamps are not used and, accordingly, it is appropriate to switch from the terminology of "leave it be" and "push the button" for, respectively, $+$ and $-$ to that of "plus" and "minus". This should occasion no problem for the student since he/she is familiar with this terminology from his/her experience with natural number arithmetic.

Activity 4 is a rather easy activity and the students should dominate it fairly quickly, but it does serve as a useful transition from an empirical based context to a more abstract arithmetical context, as well as a transition from the empirical terminology to arithmetical terminology. The next activity helps to consolidate this transition.

Activity 5

The players form teams of two or three members. A partial template, like that of Figure 7 is presented to them and they are tasked with presenting all the combinations of operations that make true equivalences. The team which lists all the possibilities in the least amount of time wins.

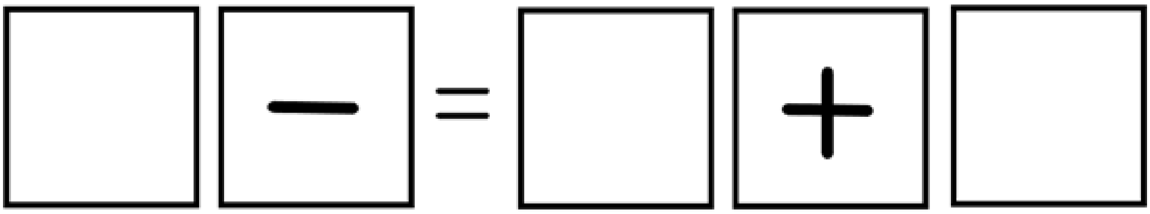
Figure 7. An open equivalence



Source: Elaborated by the author

There is no need to initiate play with the first two cards as the givens, as in Figure 7. Thus, a configuration like that of Figure 8 is also possible.

Figure 8 – An open equivalence



Source: Elaborated by the author

By this time the student should be ready to handle double negatives without much difficulty. Should that not be the case either for all or for some of the students, the next activity can be presented to those who need it.

Activity 6

This is just the lamp activities with numbers in the place of lamps and arithmetical terminology in place of empirical terminology. Discarding occurs by forming true arithmetical statements.

Auxiliary Problems

As the class proceeds to the arithmetical context, the teacher should keep in mind that arithmetical errors in negative number arithmetic may be due, not to the negative numbers themselves, but to other insufficiently apprehended procedures. The distributive law is a common problem, as many students write, for example, the numerical equivalent of

$$a(b+c) = ab+c.$$

Sometimes this is just inattentiveness, but generally it is due the correct rule not being internalized. Naturally, these problems will have to be addressed in other ways.

CONCLUSION

Although not occurring in the same manner as it did historically, the introduction of negative numbers in school arithmetic is nonetheless an expansion of the student's concept of number. In expanding this concept, it also becomes necessary to generalize the arithmetical operations in order to operate with the new kind of number introduced. Both of these two generalizations may cause cognitive obstacles of one type or another. Indeed, the acceptance of negative numbers by mathematicians was historically problematic. It is not generally troublesome, however, for the contemporary student due to the presence of these numbers in his/her cognitive base before he/she encounters them in the school context.

With regard to the arithmetical operations, a didactical obstacle may arise in subtraction (addition), whenever the subtrahend is greater than the minuend, due to prior instruction that negated the possibility of "taking the greater from the lesser". This problem can usually be overcome by using the number line as a visual/conceptual support for addition and subtraction. Multiplication, in contrast, generates a genuine epistemological obstacle since the multiplication of two negative numbers, resulting in a positive number, makes but little sense, both historically and pedagogically, due to the fact that the usual metaphors used to explain negative numbers are befuddling in the context of multiplying negatives.

It was proposed herein that considering the negative numbers as (additive) inverses of their positive counterparts could go a long way towards overcoming this obstacle because the inverse of the inverse is a familiar notion. Accordingly, a series of activities was proposed, in which double negatives are approached as inverses of inverses. The empirical activity of turning on and off a lamp, associated with the use of the minus sign to indicate a change of state, strengthens the student's intuitive knowledge of the inverse of the inverse and gradually transfers this intuitive awareness to the arithmetic of negative numbers. Naturally, the proposal still needs empirical testing with regard to possible modifications, sequencing, and efficacy.

REFERENCES

BACHELARD, Gaston. **La formation de l'esprit scientifique**. Paris: J. Vrin, 1938.

BARLOW, Peter. **A New Mathematical and Philosophical Dictionary**. London: G. and S Robinson, 1814.

EUCLID. **The Thirteen Books of the Elements**. Trans. Thomas L. Heath. 3 vols. New York: Dover, 1956.

FOSSA, John A. **Os Primórdios da Teoria dos Números**. 2 v. Natal: Editora da UFRN, 2010.

FOSSA, John A.; ANJOS, Marta Figueredo dos. Sobre a incompatibilidade dos números negativos com o conceito grego de *årithmós*. **Revista Brasileira de História da Matemática**, Rio Claro, SP, v. 7, n. 14, p. 163-171, 2007.

KUHN, Thomas. **The Structure of Scientific Revolutions**. Chicago: University of Chicago Press, 1962.

LUMPKIN, Beatrice. Zero in ancient Egypt? 2005. Disponível em: http://www. ethnomath. org/resources/ISGEm/084.htm. Acesso em 23/06/05.

MENDES, Iran Abreu; FOSSA, John A.; VALDÉS, Juan E. Nápoles. **A História como um agente de cognição na Educação Matemática.** Porto Alegre: Sulina, 2006.

PEACOCK, George. **Treatise on Algebra**. 2 vols. Mineola (NY): Dover, 2004.

PYCIOR, Helena M. **Symbols, Impossible Numbers and Geometric Entanglements**.

Cambridge: University of Cambridge Press, 1997.

SCHUBRING, Gert. **Números negativos:** exemplos de obstáculos epistemológicos? São Paulo: Livraria da Física, 2018.

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