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Discovering geometry via the Discover command in GeoGebra Discovery

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ABSTRACT

We present a new way to discover statements in a planar geometric figure by using GeoGebra Discovery, an experimental version of GeoGebra, the free dynamic mathematics software package. A new command "Discover" (which is also available as a tool) requires an input point of the figure---as output several properties of the figure are communicated by the program. That is, "Discover" reports a list of the observed geometric properties, including point equality, equal long segments, collinearity, concyclicity, parallelism and perpendicularity. All of the obtained statements are checked symbolically: this means that the verification is done with computer algebra means. The obtained properties are also highlighted with colors or dashed lines in the original figure. The discovery process can always be continued by creating new objects and selecting a new target point to discover. We focus on possible uses in a classroom: two basic examples are shown from an Austrian textbook first. Then some more difficult topics are introduced that are usually covered by the secondary school curriculum. As a final example, we consider the discovery of a more advanced theorem, namely, a proposition according to Napoleon. In the paper we give some references to related software systems and the applied mathematical background as well.

Keywords: GeoGebra; Discovery; Planar Geometry, Proving Statements.

Descobrindo a geometria através do comando Discover no GeoGebra Discovery

RESUMO

Apresentamos uma nova maneira de descobrir afirmações em uma figura geométrica plana usando o GeoGebra Discovery, uma versão experimental do GeoGebra, o pacote de software de matemática dinâmica gratuito. Um novo comando "Discover" (que também está disponível como uma ferramenta) requer um ponto de entrada da figura --- como saída, várias propriedades da figura são comunicadas pelo programa. Ou seja, "Descobrir" relata uma lista das propriedades geométricas observadas, incluindo igualdade de pontos, segmentos longos iguais, colinearidade, conciclicidade, paralelismo e perpendicularidade. Todas as afirmações obtidas são verificadas simbolicamente: isso significa que a verificação é feita com meios de álgebra computacional. As propriedades obtidas também são destacadas com cores ou linhas tracejadas na figura original. O processo de descoberta sempre pode continuar criando novos objetos e selecionando um novo ponto de destino para descobrir. Nós nos concentramos em possíveis usos em uma sala de aula: dois exemplos básicos são mostrados primeiro em um livro austríaco. Em seguida, são introduzidos alguns tópicos mais difíceis que geralmente são cobertos pelo currículo da escola secundária. Como exemplo final, consideramos a descoberta de um teorema mais avançado, a saber, uma proposição de acordo com Napoleão. No artigo, fornecemos algumas referências a sistemas de software relacionados e também à base matemática aplicada.

Palavras-chave: GeoGebra; Descoberta; Geometria plana, declarações de prova.

Descubriendo geometría usando el comando Descubrir en GeoGebra Discovery

RESUMEN

Presentamos una nueva forma de descubrir afirmaciones en una figura geométrica plana mediante el uso de GeoGebra Discovery, una versión experimental de GeoGebra, el paquete de software gratuito de matemáticas dinámicas. Un nuevo comando "Descubrir" (que también está disponible como herramienta) requiere un punto de entrada de la figura --- como salida, el programa comunica varias propiedades de la figura. Es decir, "Discover" informa una lista de las propiedades geométricas observadas, incluida la igualdad de puntos, segmentos largos iguales, colinealidad, conciclicidad, paralelismo y perpendicularidad. Todos los enunciados obtenidos se verifican simbólicamente: esto significa que la verificación se realiza con medios de álgebra computarizada. Las propiedades obtenidas también se resaltan con colores o líneas discontinuas en la figura original. El proceso de

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descubrimiento siempre puede continuar creando nuevos objetos y seleccionando un nuevo punto de destino para descubrir. Nos centramos en los posibles usos en un aula: primero se muestran dos ejemplos básicos de un libro de texto austriaco. Luego, se introducen algunos temas más difíciles que normalmente se tratan en el plan de estudios de la escuela secundaria. Como ejemplo final, consideramos el descubrimiento de un teorema más avanzado, a saber, una proposición según Napoleón. En el artículo damos algunas referencias a sistemas de software relacionados y también a los antecedentes matemáticos aplicados.

Palabras clave: GeoGebra; Descubrimiento; Geometría plana, declaraciones de prueba.

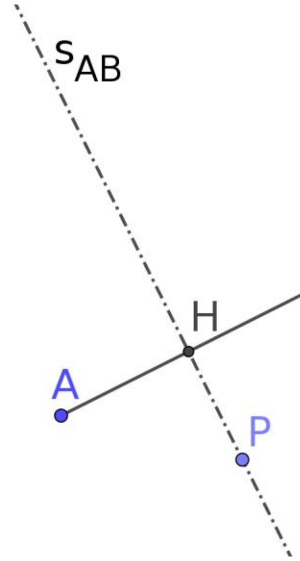
INTRODUCTION

Geometry in teaching mathematics has a changing role in nowadays' schools. In various countries there is different tradition of teaching geometry, and even in the same country there can be different concepts within the same generation.

Without any doubt geometry is one of the oldest branches of mathematics. Its spatial character distinguishes it from other branches, and it is still mind-challenging to explain generality of statements. For example, the Pythagorean theorem claims that for all points A, B and C in the plane, assuming $AB \perp BC$, for the lengths of AB, BC and AC the equality $AB^2 + BC^2 = AC^2$ holds. In a large number of countries this knowledge is expected to be well-known for all young learners, not just because of its historical importance but its benefit present already in the simplest computations.

Actually, the Pythagorean theorem has an algebraic character, unless we consider it as a sum of areas. On the other hand, in this paper we focus on statements that have a more pure geometric character. One very simple example is that a perpendicular bisector of a segment contains the set of all points that are equally far from the end points of the segment. Here equality can be approached, of course, also from the algebraic point of view, but equality of lengths can be explained as a congruence which is a geometric notion. In Figure 1 we can see that each point P that is an element of the perpendicular bisector sAB yields the equality $AP = BP$.

Figure 1 – sAB is the perpendicular bisector of AB



Source: Elaborated by the author

In the following we introduce an experimental version of the well-known GeoGebra software package (HOHENWARTER, 2002). It includes a new command and tool, namely Discover. This extension considers the current figure given in GeoGebra as its input and a point

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that is selected by the user. Then the Discover command collects as much useful information in the figure as possible. These pieces of information are all verified facts that are true in general (or, in other words, not considering a couple of counterexamples, essentially always, see (KOVÁCS; YU, 2020) for a more precise explanation, or true on parts, that is, usually because of algebraic symmetry there can be some cases where the facts are not true, but under some additional conditions the statement will be generally true, see (KOVÁCS; RECIO; VÉLEZ, 2019), and not just a numerical check is performed.

Therefore, a conjecture collected by the computer is always verified by computer algebra means, roughly speaking, by turning the geometric figure into an equation system and manipulating on it with advanced algorithms in algebraic geometry, See Chou (1988) and Cox, Little and O'Shea (1991), chapter 6, for a detailed introduction to the theory with examples. In Fig. 2 and 3 this concept is shown, by using Fig. 1 as the source of investigations. The equal long segments are identified and colored with the same color (green), but also the non-trivial perpendicular property is highlighted in red.

Figure 2 – Report on the discovered theorems of Fig. 1

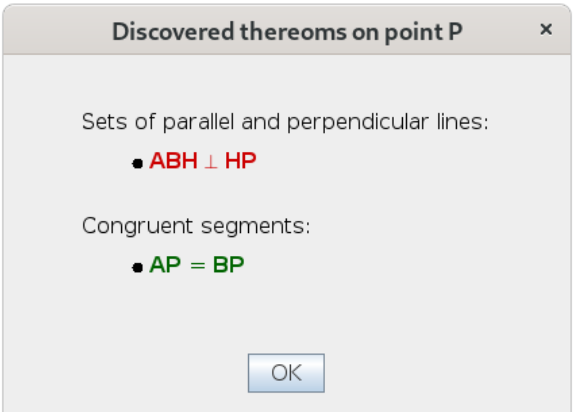

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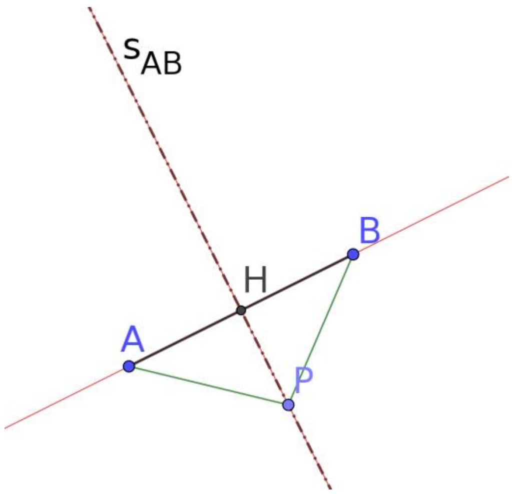
Figure 3 – Visualization of the discovered theorems



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We emphasize here that Fig. 1 was drawn by copying an assignment from a school textbook Humenberger, Litschauer, Groß and Aue (2013, p. 57, E, H4/2) written for 14-years-old students, actively used in Austria at present day.

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In this paper we would like to illustrate that this new feature can be used as a tool that can improve teaching and promote self-experimenting and therefore understanding, on the student's side. We admit that currently there is no statistical analysis if this novel tool is indeed useful in schools. We leave this question for further research.

Also, we refer the reader to further literature where there are precursors of the current research. We emphasize that the novelty of our work is adding the Discover command to GeoGebra, a software tool that is used by several millions of students and teachers; therefore, a clear outcome of the present research is improved understanding of planar geometry for a much larger amount of learners than before.

THE PROTOCOL

First of all, the user requires an experimental version of GeoGebra that is available for download at http://github.com/kovzol/geogebra-discovery. We describe here the version 2020Oct23. Alternatively, the web page http://autgeo.online provides an online version with the same functionality. (Some features are, however, usually slower, because the computer algebra computations perform not as quick in a web browser as natively.)

After sketching up the figure, the Discover tool (or command) must be issued by selecting a point in the figure (or typing the command **Discover(P)** in the Input Bar, here P denotes the point to observe). We illustrate the process by taking another assignment from Humenberger, Litschauer, Groß and Aue (2013, p. 58, E, H4/5). The student needs to investigate the distance between the intersection point of the angle bisectors of three lines in the textbook assignment. In Fig. 4 and 5 we can see the construction made with GeoGebra Discovery (three lines: p, q and r, two angle bisectors of them and their intersection X) and the result of the discovery process (after the user clicks on the Discover icon  and selects point X). This first attempt gives no result: the student needs to define some further points that will introduce something remarkable. (Note that we denoted the intersection points of lines p, q and r for further reference.)

Figure 4 – Intersection of angle bisectors

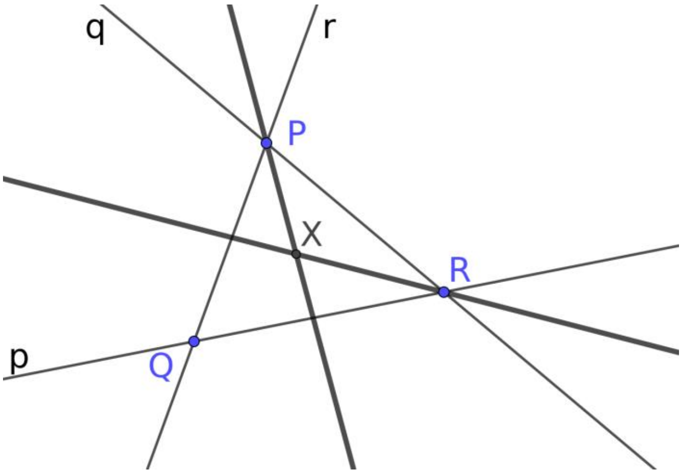

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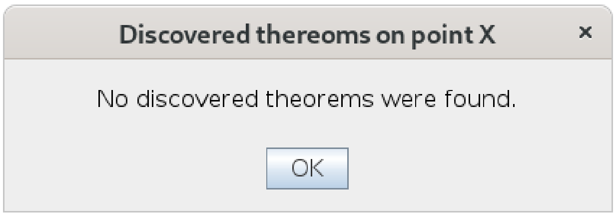
Figure 5 – Unsuccessful discovery



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For this reason, two perpendiculars from X to the lines p and r will be created, and then their intersections D and E with the lines p and r, respectively (Fig. 6). At this stage we can already conclude some interesting features of the construction (Fig. 7). The most expected property is the equality $AX = BX$, but some, maybe unexpected features can also be found.

Figure 6 – Extending the figure with two foot points of X

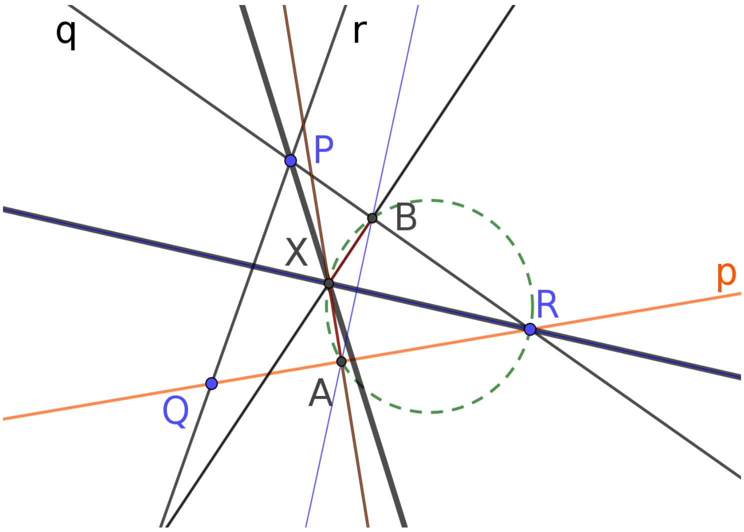

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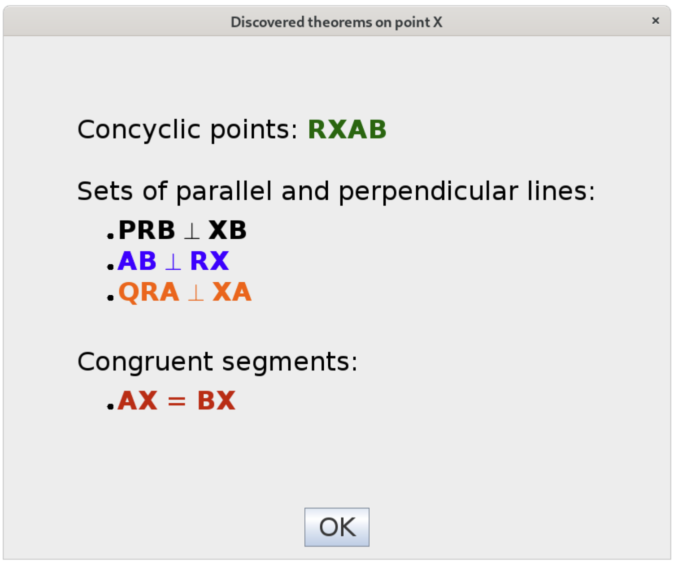
Figure 7 – Successful discovery



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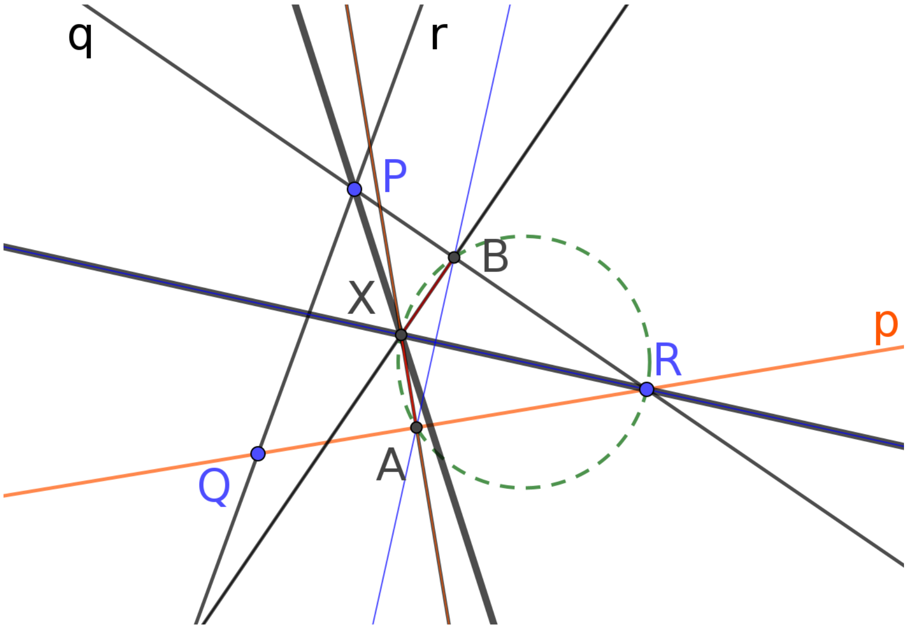
These results are visualized in Fig. 8. Concyclic points Q, X, A and B are connected with a dashed circle, receiving a green color for this presentation. Perpendicular lines usually receive the same color, here AB and RX are highlighted with blue. (Actually, the other two perpendicular properties can be considered trivial.)

Figure 8 – Visual report of Discovery



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Further discovery can be obtained when another point C is introduced. In Fig. 9 we project point X on line r and obtain C, then re-run discovery for point X. So we get Fig. 10 and 11.

Figure 9 – Introducing foot point C

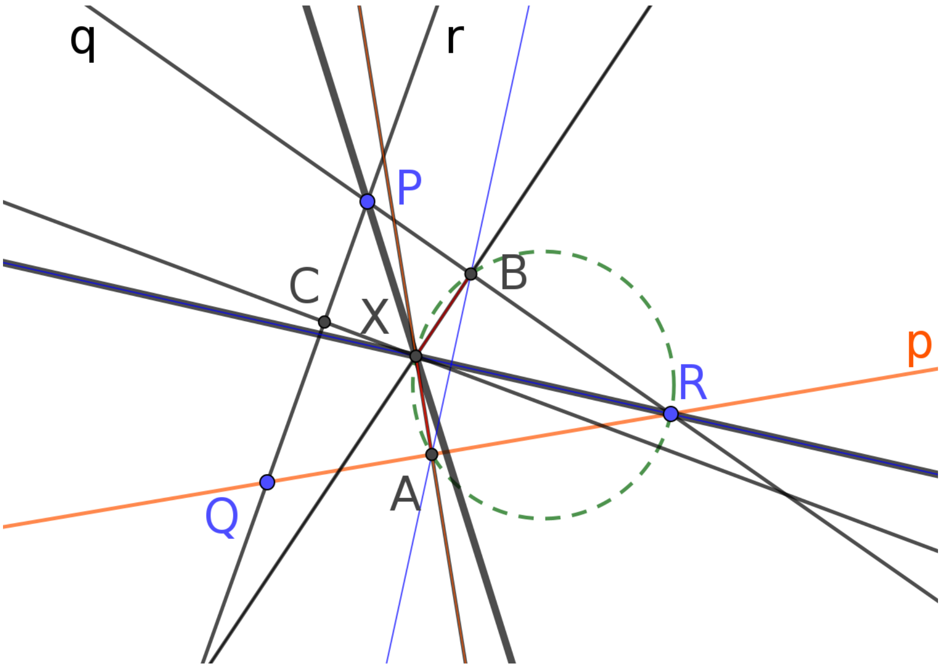

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Figure 10 – Extended discovery

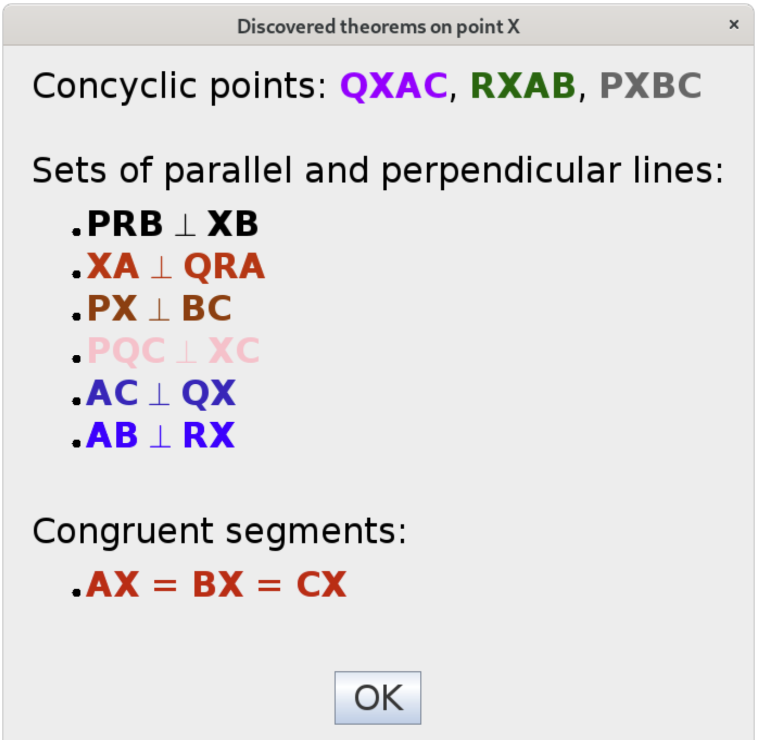

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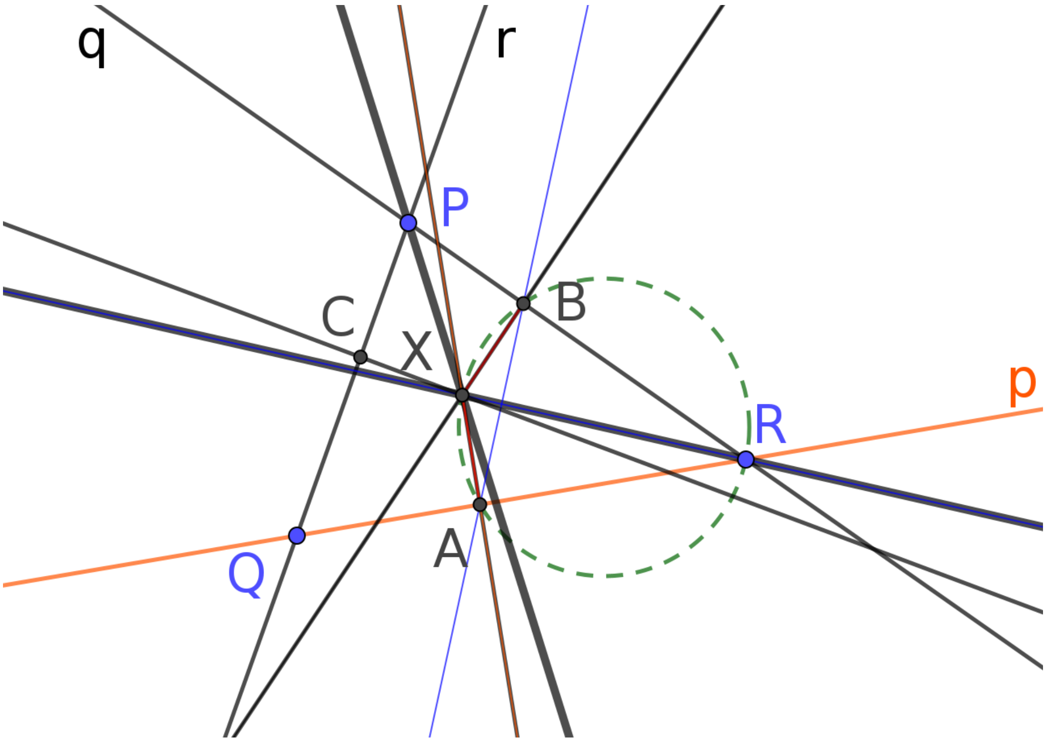
Figure 11 – Visual explanation of extended discovery



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The shown output is usually a set of refined and filtered data. We highlight that lines PRB, QRA and PQC are handled as unique objects, just like circles $QXAC$, $RXAB$ and $PXBC$. In general, congruent segments are also considered as sets of equal long objects. Collecting these facts in such a convenient way may be trivial for a human, but for a computer not: sophisticated algorithms are required in the background to keep the data consistent and to obtain them as quickly as possible.

SOME MORE EXAMPLES

From the secondary school curriculum we can mention a couple of examples where the Discover command in GeoGebra Discovery can be a handy tool for the learners.

These include *Thales' circle theorem*, some properties of a regular octagon, or *Euler's line* (that is, the line that connects the orthocenter, the circumcenter and the centroid of a triangle: they are collinear unless the triangle is equilateral), illustrated respectively in Fig. 12, 13 and 14.

Figure 12 – Thales' circle theorem: Discovery about point *D*

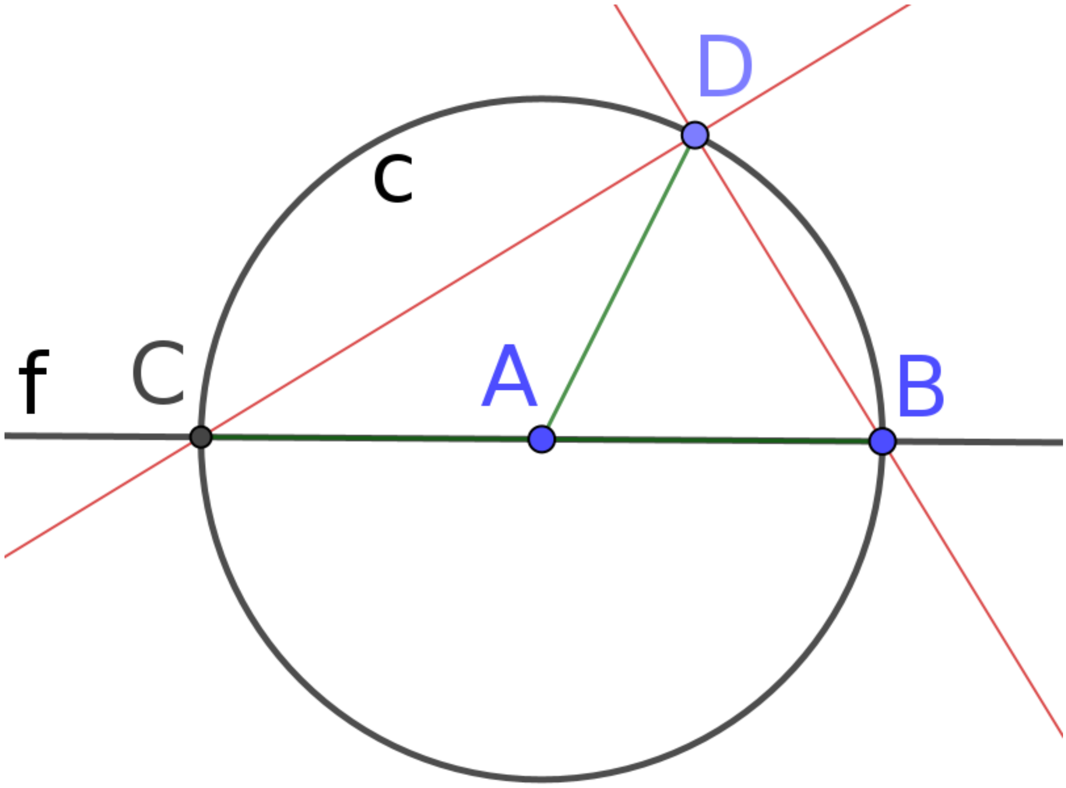

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Figure 13 – Properties in a regular octagon

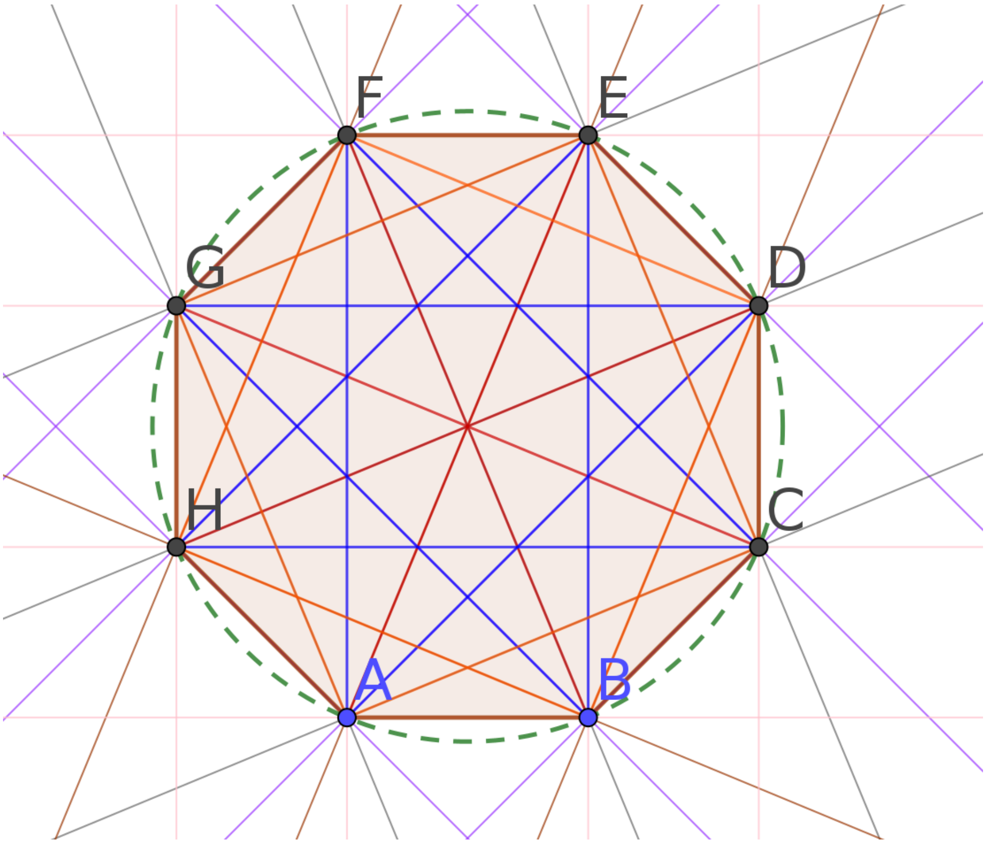

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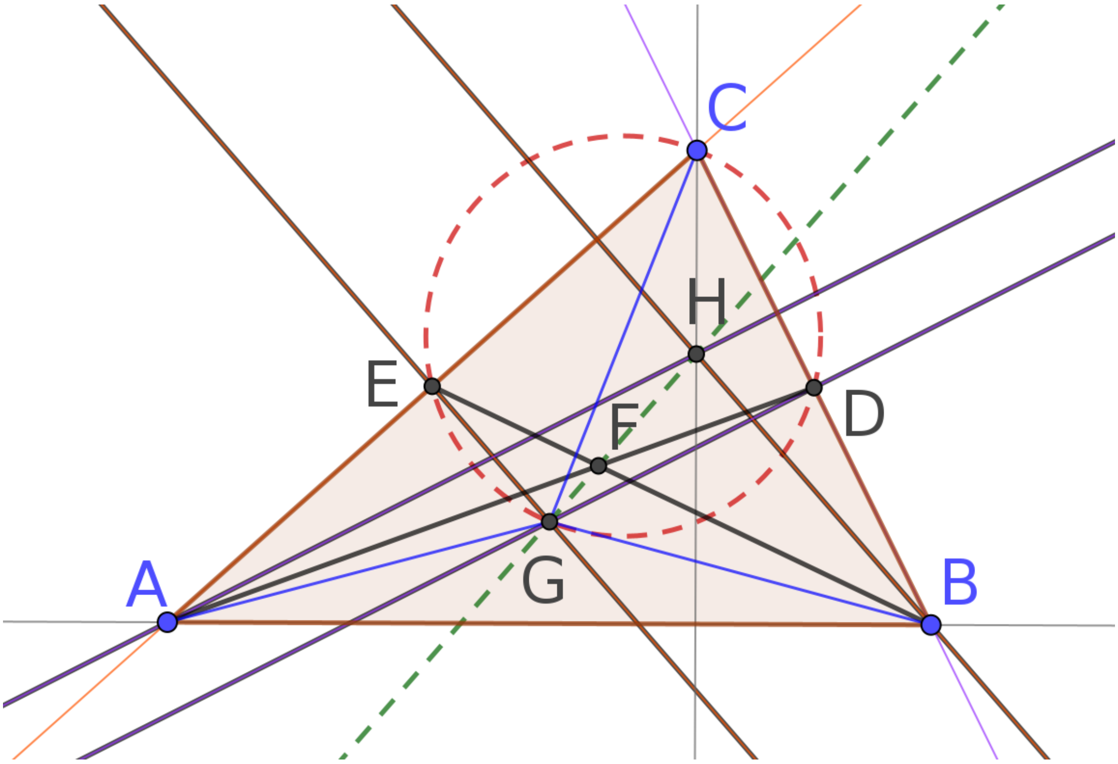
Figure 14 – Euler's line: Discovery at point *G*



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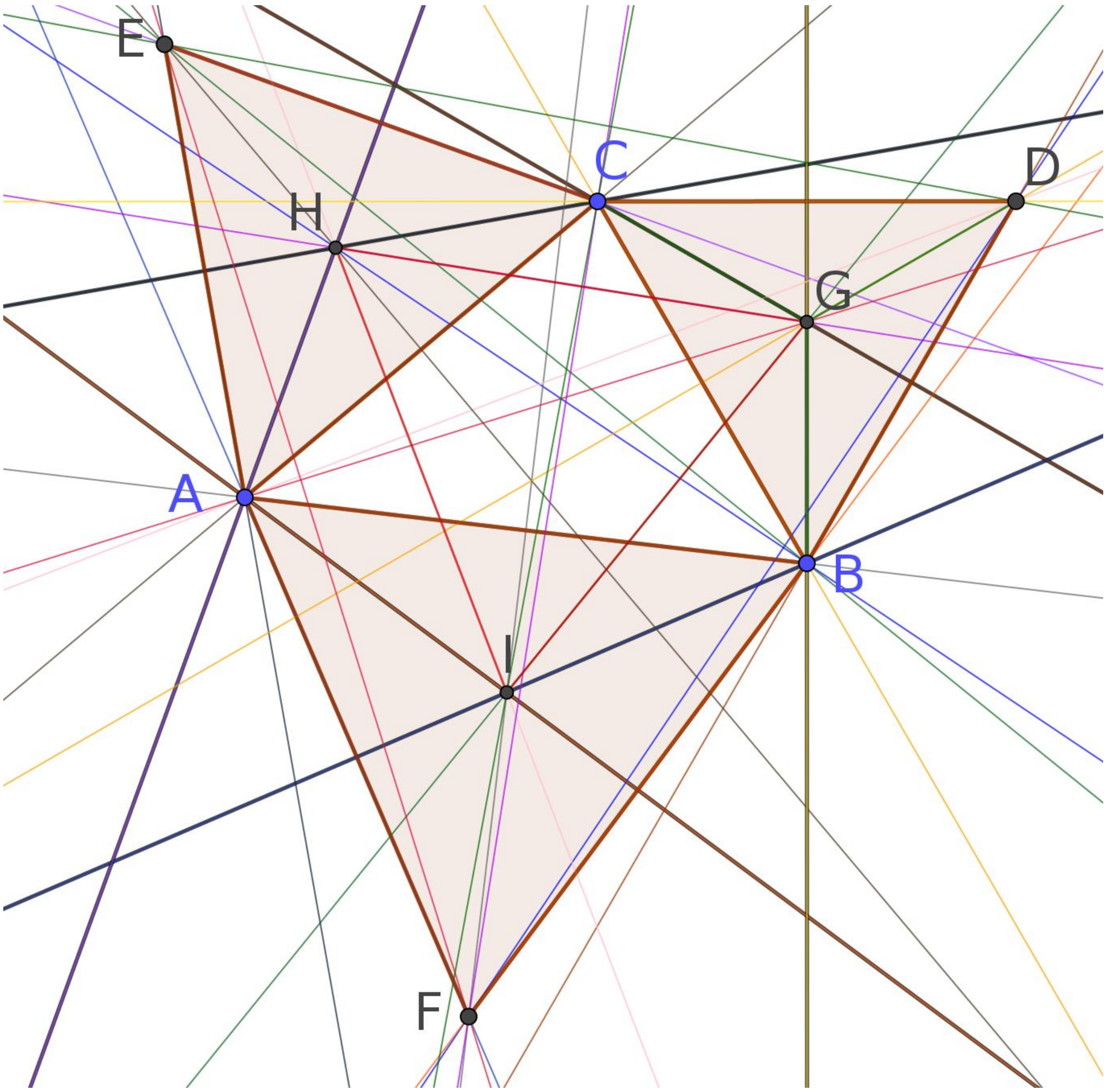
We highlight that the obtained properties are all verified facts. Verification means that the property is indeed true in general, or on parts.

Gifted learners or classes in higher-level education may benefit from further experimenting. As a closing example we demonstrate the Discover tool by obtaining Napoleon's theorem after sketching an arbitrary triangle ABC and erecting regular triangles outside on its sides. Then, by using intersections of perpendicular bisectors we create midpoints G, H and I for the erected triangles. Discovery is started on point G. Fig. 15 shows the result. It includes not just the equality $GI = GH = HI$, but some, maybe surprising facts that CI and DE (or BE and GI) are perpendicular.

Figure 15 – Napoleon's theorem



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We can summarize that discovery supports self-experimenting and can lead to unexpected results. But this is a fully natural character of mathematics, and as such, it deserves emphasizing at all levels in the education.

RELATED WORK

There are several precursors of the presented work. In the non-symbolic literature we refer to (MAGAJNA, 2011) that exemplifies a software tool that is capable to obtain thousands of interesting features in a planar geometric figure. It works, however, by using numerical algorithms in the background.

In the symbolic literature we need to mention the first mechanical experiments on obtaining a proof by using Wu's algebrization strategy, see Wu (1978) and Chou (1988). Similarly to that concept, the method we use in the background requires heavy computer algebra.

GeoGebra Discovery's experiments are closely related to the first experiences of the development of Automated Geometer (BOTANA; KOVÁCS; RECIO, 2018), a web application that uses GeoGebra for its backend computations. By contrast, the Discover tool in GeoGebra Discovery is an embedded subsystem in the GeoGebra infrastructure inside, and has many technical novelties, for example, to maintain the database of geometrical objects in classes of equivalence relations, see Kovács and Yu (2020) for more details.

CONCLUSION

We showed an experimental version of GeoGebra that is able to analyze the input planar geometric figure and obtain several non-trivial properties of it in a symbolic way. As a result, the obtained relations are checked if they are true in general (or on parts) immediately. This may help understanding the difference between conjecture and mathematical truth for tomorrow's learners. Also, gifted students or teachers can start their own experiments in a well-known software infrastructure like GeoGebra. Being open sourced, this technology is available for millions of students worldwide free of charge.

We admit that the shown method is still subject to improve. In some cases, speed should be further optimized: this is especially true for GeoGebra Discovery's web version. In some cases the trivial properties should be hidden to avoid leading the user into confusion: this can happen if the number of collected properties is high. Also, some user interface polishment (for example, including a Cancel button when the computations take too long) is a next step for the development.

Acknowledgments

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