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On the Hermeneutics of Reading Historical Texts in the Mathematics Classroom

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ABSTRACT

A hermeneutical substructure for the reading of historical texts in the mathematics classroom is described. It is organized according to the four transcendentals of unity, truth, beauty and goodness, each of which is considered internally –with respect to the text itself – and externally – with respect to both its relations to mathematical thought in general and other cultural manifestations. Lead questions are provided for each category as are also details regarding various ways in which the student's interactions with the text can be organized in order to provide the student with a rich learning experience. Some attention is also given to questions relating to implementation.
Keywords: History of Mathematics in Mathematics Education; Hermeneutics; Reading historical texts.

Sobre a Hermenêutica da Leitura de Textos Históricos na Sala de Aula de Matemática

RESUMO

Descreve-se uma estrutura hermenêutica para a leitura de textos históricos na sala de aula de matemática. É organizada pelos quatro transcendentais de unidade, verdade, o belo e o bem, cada um dos quais é considerado internamente – com respeito do próprio texto – e externamente – com respeito às suas relações tanto com o pensamento matemático em geral, quanto com outras manifestações culturais. As perguntas principais são dadas para cada categoria, bem como detalhes sobre várias maneiras em que as interações do aluno com o texto podem ser organizadas a fim de propiciar ao aluno uma rica experiência educacional. Alguma atenção é também dada a questões de implementação.
Palavras-chave: História da Matemática em Educação Matemática; Hermenêutica; Lendo textos históricos.

Sobre la Hermenéutica de la Lectura de Textos Históricos en el Aula de Matemáticas

RESUMEN

Se describe un marco hermenéutico para la lectura de textos históricos en el aula de matemáticas. Está organizado por los cuatro trascendentales de la unidad, la verdad, lo bello y lo bueno, cada uno de los cuales se considera internamente – con respecto al texto mismo – y externamente – con respecto a sus relaciones tanto con el pensamiento matemático en general como con otras manifestaciones culturales. Se dan preguntas clave para cada categoría, así como detalles sobre las diversas formas en que se pueden organizar las interacciones de los estudiantes con el texto para brindarles una rica experiencia educativa. También se presta cierta atención a los problemas de implementación.
Palabras clave: Historia de las Matemáticas en la Educación Matemática; Hermenéutica; Lectura de textos históricos.

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INTRODUCTION

The reading of historical texts in the mathematics classroom can be an effective pedagogical strategy for the development of both mathematical knowledge and mathematical skills.[2] It is equally effective in the context of disciplines directed towards the history of mathematics and those focusing on the content of mathematics, even though the pedagogical objectives are clearly somewhat different for the two types of classroom settings. As an exercise in reading, the skills needed to come to terms with the text are not very distinct from those needed to interpret, for example, literary works;[3] nevertheless, the specific content of mathematical texts does require some special knowledge and skills pertinent to the content. Just as in the case of literary texts, however, the most effective way of attaining the necessary skills is the "bootstrapping technique" of reading, or better, of guided reading under the tutelage of a more experienced teacher. Since many mathematics teachers have not themselves developed the aforementioned skills, the present article presents a checklist, as it were, of the kinds of dialogue that the reader can have with the text.

It will be well to observe that the suggestions made herein pretend to attain to a certain hermeneutical completeness and, thus, it is not proffered as a strict recipe to be implemented in its entirety in all circumstances. Rather, it may be used as a brief hermeneutical handbook from which appropriate directives can be adopted, depending on the objectives of the reading to be undertaken at any given moment, although it is to be expected that students of the history of mathematics would utilize a more comprehensive portion of the suggestions than those in mathematics content courses for the simple reason that the history of mathematics has a richer and more diverse set of objectives than that of mathematics content courses.

Before proceeding further, it will also be well to remark on the meaning of "mathematics". Elsewhere[4], the present author has given two different definitions of this word. Herein, however, I will consider it to be a primitive term, although I will make some additional observations about this in the section (below) regarding external goodness.

THE HERMENEUTICAL SUBSTRUCTURE

In order to obtain a reasonably complete framework for supporting the student's interpretive efforts, I have devised a hermeneutical substructure based on the theory of the four transcendentals. The theory was a staple in medieval philosophy; its basic tenant is that certain categories are so general as to apply to all beings and, thus, are instrumental to the most fundamental description of being. Naturally, there was robust debate about which categories were to be considered transcendental and even as to the number of the transcendentals. Herein I am not interested in defending the theory of transcendentals, nor am I concerned to defend my choice of categories. Rather, I only wish to present my choice as a convenient way of codifying,

[2] See, for example, Fossa (2020).
[3] There have been some studies, for example Thomas (2000), that trace analogies between doing mathematics and storytelling. The parallels to which we will be drawing attention throughout the present article, however, are those that result from the similar hermeneutical enterprises of reading mathematical and literary texts.
[4] See, for example, Fossa (2004) e Fossa (to appear).

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organizing and cataloging a large number of directives that teachers may find useful in accompanying their students' adventures in reading historical mathematics texts.

Each of the four transcendentals – unity, truth, beauty and the good– can be applied to aspects and relations of the text itself, considered as the nexus of a dialogue between the author and the reader, as well as to relations that it maintains, as always through the lens of the reader, with other cultural constructs.[5] Consequently, we obtain the following eightfold structure:

Internal Unity	External Unity
Internal Truth	External Truth
Internal Beauty	External Beauty
Internal Goodness	External Goodness.

I will explain, presently, each of these elements and how each relates to reading historical mathematics texts in the classroom. Before doing so, however, it is incumbent on me to observe that the transcendental *unity* is usually called "being" (in contrast to "Being"). Nevertheless, the meaning of beings (that is, the beings in the universe) is conceived of as unities and, moreover, the term "unity" is more evocative of the application to which it will be put herein. The other three terms are not problematical.

INTERNAL UNITY

Once again, we may avail ourselves of an analogy with literary texts. It is expected that, say, a short story have a certain theme, be it that of telling a coherent story, presenting a character sketch, *etc*. In fact, ancient literary critics believed (following Aristotle's *Poetics*) that theatrical tragedies should even have a certain unity of time and place. Likewise, we expect that mathematical texts also tell a unified story. The story may be rather loose, as, for example, a text on the Calculus, or more tightly knit, such as a treatise on continuity in topological spaces. Even in the former case, however, the inclusion of extraneous material is frowned upon; thus, if some geometrical theorems are included in a text on the Calculus, it is only because they further the resolution of some Calculus problems.

Naturally, the degree of unity exhibited by any text *may* be affected by the nature of the text itself. A book may contain much more disparate information than a journal article or a chapter of a book. Further, a very common practice in using historical texts in the classroom is that of not presenting the whole text, but only an excerpt (or excerpts) and this may distort the student's judgements regarding unity. In any case, the student has to come to grips with whatever text he is presented with (just as the historian of mathematics is sometimes only in

[5] This division is clearly related to, though not the same as, Jankvist's distinction between "history as a tool" and "history as a goal" with reference to why original texts should be adopted in the mathematics classroom. Cf. Jankvist (2009).

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possession of partial texts), although the teacher should provide some background information regarding the provenance of the adopted excerpts.

Another factor that can affect the unity of any mathematical text is that of the author's objectives in writing the work. Thus, whenever possible, the student should identify the author's objectives and verify whether they have been adequately fulfilled in the text. This is part of the evaluation of the internal unity of the work. It may happen that the identification of the author's objectives requires some extra-textual detective work and it will be up to the teacher to decide whether and/or how much of this kind of investigation is to be undertaken by the student. It may also happen that some of the author's objectives may seem utterly irrelevant. S(he) may need book sales, for example, to shore up her (his) failing financial situation and this may, or may not, impinge on our judgement regarding the book's unity. In any case, we may distinguish between the *work's objectives* and the *author's objectives*, the former being a subset of the latter. One should always try to determine the work's objectives.

The determination of the work's objectives is often revealed by theme statements in the work itself. Other evidence is often found in the title of the work and subtitles, if there be any, of various sections and subsections. There may also be summaries, most likely in the beginning of the text, either as an abstract or in the introduction, and at the end of the text, in the conclusion, though there may also be partial summaries dispersed in other parts of the text. One should proceed with caution, however, since historical texts, especially excerpted historical texts, may have titles/subtitles and/or summaries provided by a later editor. The editor's additions may, or may not, reflect the author's intentions.

Another aspect of internal unity is the independence of the work, that is, is the presentation self-contained so that it can stand on its own or does the reader have to have recourse to other writings in order to read the work with understanding? Is all of the symbolism explained? Are all relevant definitions presented? Are all the required prior results present or, at least, sufficiently referenced? All of these are important considerations, but we should also take into account the fact that mathematics is both a collaborative and a culminative enterprise, so that we can hardly expect each mathematician to start over *ab ovo*. The correct procedure would seem to be to ascertain what background knowledge the author could reasonably expect from his intended audience and, with this in mind, to evaluate the text with regard to any lacunae. The resulting appraisal need not be absolute (it could contain such phrases as "it might have been helpful …"). The indicated procedure, however, would probably be too ambitious for many classroom settings, even with the help of the teacher. In any case, the student should try to locate any missing links that prejudices his (her) own understanding of the text.

EXTERNAL UNITY

Whereas judgements regarding internal unity assess the text as an independent, unified being, those regarding external unity take account of the already mentioned facts that mathematics is a collaborative and cumulative activity undertaken by a community of mathematicians. Thus, it adopts a predominately diachronic perspective, in contrast to the more synchronic perspective of internal unity, and attempts to fit the given work into the historical flux of mathematical thought. Considerations such as these will, therefore, be slightly more

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important to teachers involved in history of mathematics courses, rather than those who are interested in using historical texts as aids in, for example, concept formation.

One of the most important determinations regarding the external unity of a mathematical work is that of describing its relation to the state of prior mathematics. Does it solve outstanding problems and/or advance the understanding of prior theoretical perspectives? Does it reveal an underlying unity in apparently disparate approaches? Or is it more retrospective, summarizing and perhaps reflecting upon the state of mathematics at the time of its being written? Or is it so innovative that it, in effect, makes a new beginning? To what degree does the new beginning indicate a rupture in the historical flux of mathematical thought and to what degree is it more akin to a restructuring of the prior flux?

Once some at least tentative conclusions regarding the text's mathematical heritage have been reached, we may turn our attention to its progeny. That is, we want to investigate its role in the development of later mathematical thought. This will be especially important for those innovative works that make new beginnings because a work that does not participate in the cumulative effort of the mathematical community stands outside of the developing mathematical culture. This, of course, does not mean that it is necessarily an unremarkable mathematical work. But is does mean that it possesses, at the present time, little external unity. The parenthetical disclaimer "at the present time" is necessary since it ofttimes happens that a work is marginalized for some time, only to be incorporated into the mathematical tradition at a later time.

Finally, we should recall that mathematics is a cultural construct of mankind and, thus, is related to many other aspects of human culture. Frequently new, even innovative, mathematics results from problems originating outside of mathematics, such as the development of graph theory from Leonhard Euler's response to the Königsberg bridge problem. Perhaps just as frequently, other cultural products stem from mathematics, such as Plato's philosophical response to certain mathematical structures. Perhaps less esoterically, it is also necessary to take into account the role of pure and applied mathematics in the mathematical modeling used in science and technology. All of these considerations redound to the given text's relation to the unity of the overarching human culture.

INTERNAL TRUTH

Truth in mathematics, from a general point of view, is consistency. Hence, the internal truth of a mathematical text is the telling of a coherent story. All of the parts of the story should "hang together". Nonetheless, the telling of a coherent mathematical story is more than just the absence of contradiction; it must also be a compelling story if it is to have mathematical validity. To see this, consider, for example, a text that consists simply of a list of unrelated propositions. If they are really unrelated (and if each of them is not self-contradictory), they will not engender any contradiction. But clearly, they also will not engender anything that can be called a "story"

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since a story implies structure.[6] Further, the compellingness (that is, the forcefulness) of a mathematical story comes from the deductive relations among the elements of the structure.[7]

There are at least three considerations that we should keep in mind in evaluating internal truth. The first is on the macro level, since it concerns the overall structure of the work itself. The basic concern at this level is to determine whether the story told actually achieves the objective of the text. That is, are the deductive relations among the propositions presented sufficient to establish the main result? The main result may be a theorem, a set of theorems or a characterization of a mathematical concept, structure or procedure. When the text is organized as a series of propositions accompanied by their demonstrations, our task is facilitated since the proposed deductive structure is more evident and our task reduces to checking that the proposed argument is indeed valid. Thus, in order to evaluate a text that proposes to demonstrate the Fundamental Theorem of Arithmetic through the use of three lemmas, we must verify that the lemmas do, in fact, entail the mentioned Fundamental Theorem. For a text that covers the same material, but which is not organized on a theorem-demonstration basis, we must first of all ferret out the strategy employed by the author, before we can verify whether that strategy contains a satisfactory deductive structure.

Certain historical texts do not have the kind of objectives that seem to be implicit in the types referred to in the previous paragraph. Reappraisals of the state of knowledge regarding a particular concept or procedure, clarifications of definitions or theorems and the posing of problems, for example, are all legitimate and important mathematical activities that may be presented in diverse fashions. Nevertheless, the overall strategic structure of the work still needs to be evaluated. These kinds of texts are more often prone to rhetorical flourishes, like appeals to the emotions; they are also susceptible to the use of unwarranted, be they tacit or explicit, presuppositions. The whole controversy over the legitimacy of negative numbers, for example, was fueled by the quite explicit presupposition that a negative number was a negative quantity. Thus, in evaluating the inner truth of these texts, we have to take into account their reasonableness, given this presupposition, as well as the reasonableness of holding the presupposition in the first place.

The other two considerations that should be kept in mind when evaluating inner truth occur at the micro level. The first of these considerations parallels, on the micro level, the macro level concern regarding the deductive structure of the whole work. That is, the deductive structure of each proposition's demonstration should be examined. It is, of course, not enough to simply determine whether each line of the demonstration being examined is suitably justified. Rather, we would also like to determine the general strategy of the given demonstration, observe any peculiar sticking points and how they were overcome, as well as to take note of any unsuspected or surprising relations with other mathematical concepts that the proof may reveal. We may also ask whether the proof is a special case that can be generalized, whether it is

[6] The exception to this is when the very list is the story, as in David Hilbert's presentation, in 1900, of what he considered to be the seminal mathematical problems for the twentieth century.

[7] The discerning reader will observe that the way the elements are structured to form a compelling story impinges upon our judgements of the story's unity. Truth and unity, therefore, are not distinct and separable categories, but are irrevocably bound up with each other. The same thing happens when we throw beauty and goodness into the mix: the four transcendentals comprise an inseparable jumble that is impossible to unravel. It is somewhat like making a well-tempered sauce with oregano, thyme and basil; we know that we used each of the spices, but cannot taste them individually in the mixture. I will not, however, insist on this point in the present article.

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productive of some general procedure, whether is it explicative rather than merely probatory and, if it is an existence proof, whether it constructs an exemplar of what is being proved.

Naturally, a whole other set of questions arises should we determine that the demonstration proposed in the text is unsatisfactory. Does the inadequacy, for example, stem from a minor mistake that is amenable to a "quick fix", or is it due to a faulty strategy, necessitating a different approach? Are there conceptual (with regard to definitions) and/or material (with regard to lemmas) lacunae in the text running up to the given proof that result in unsupported affirmations? Other deliberations may be present in relation to external truth; some of them will be mentioned in the next section.

As already touched upon in the last paragraph, definitions may be conceptually inadequate for the task at hand and, thus, the second micro level consideration with regard to internal truth proposes to examine the definitions contained in the text. Actually, the adequacy of the definitions for the various proofs of the individual propositions may also be fruitfully discussed in relation to the macro level concern of the overarching strategy adopted in the text. Nevertheless, there are other questions related specifically to the definitions themselves. If appropriate, does the text base its definitions on primitive terms, or does it try to define everything? Are the definitions circular? Are the definitions possible (non-contradictory)? Are they compossible with each other? If not, do they result in significant theorems of the kind "No A is a B"? Do they derive from or create exemplars of the *definiendum*, or are they essentially non-existential?

EXTERNAL TRUTH

In contrast to internal truth, which may be conceived of as self-consistency, external truth is the coherence of the text with the various external unities in which it participates. Being a mathematician means taking part in a cultural activity dedicated to the pursuit of mathematical knowledge and necessarily involves dialogue, be it explicit or implicit, within a cultural tradition. The most important concern, therefore, in regard to the external truth of a mathematical text is the determination of how well it fits into the historical flux of the mathematical tradition. That is, how does it contribute to the furtherance of the general objective of pursuing mathematical knowledge?

In order to further the aforementioned general objective and participate in the mathematical dialogue, one would generally expect that a text would be localizable at a historical point at which it accepts a certain body of mathematical lore, including various theorems and definitions, and, based on this acceptance, attempts to contribute to the resolution of certain agreed upon open problems (in a wide sense of this word). Indeed, when we can view a text as being at the inflection point of its mathematical heritage and progeny, it is ofttimes important to see just how much its theorems and definitions are consonant with those of its predecessors and successors.

We must be careful, however, not to make external truth such an intellectual straightjacket that it obfuscates creativity, one of the most endearing characteristics of many mathematical texts. In fact, creativity generally implies, to a greater or lesser degree, a rupture with the contemporary tradition and even the making of a new beginning. To understand the

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role that creativity plays in mathematics, we would make two observations. First, even very radical innovations do not occasion a complete break with the mathematical tradition (for, if they did, they would no longer be mathematics), so we should always be able to find connections with the mathematical tradition at some level. Second, there seems to be an asymmetry in the importance of a text's coherence with what came before it and what comes from it: progeny takes precedence over heritage. Thus, when we discern the (possibly) various points at which the text breaks with prior mathematical thought, we should ask whether the breaks were justified. From a historical vantage point, the justification of radical innovations is bound up with the proficuity of its progeny and, thus, we should try to ascertain how the mathematical tradition was affected by the work under consideration.

Another clue as to the adequacy of the text's external truth is its reception at its time of publication. Since standards of rigor, for example, are historically malleable it would be useful to determine whether any inadequacy that we have detected from the point of view of present-day mathematics would be regarded as an inadequacy by the author's contemporaries and, if not, what, exactly, has changed in the meantime. In general, were the methods and procedures used in the text acceptable to the author's contemporaries and were any implicit presuppositions of the text shared by the mathematical tradition at the time of the text's publication. Or, looking to the obverse of the coin, what was the effect of the text's revelation, should that be the case, of any implicit presuppositions in the mathematical thought of the time?

INTERNAL BEAUTY

Internal beauty may be profitably considered as the harmonious relations of its subparts. In this sense, it is a property of the structural coherence and perspicacious organization of the text as a whole. We should ask whether the text flows naturally from each subpart to others and how the disposition of each subpart illuminates other subparts of the text. In particular, does the organization of the text throw the main points into relief, highlight innovations in an attractive manner and depict open questions, especially new open questions resulting from the text itself, in an inviting way? Also, many aspects of the internal beauty of mathematical texts have to do with simplicity. Does the overall structure of the work present its theme with strong clear lines, avoiding unnecessary complexities, so that the internal relations, especially the deductive relations, of the subparts to each other are easily seen? Does (or could have) the judicious use of lemmas and corollaries contribute to the structural organization in a pleasing manner?

Again, each subpart can be subjected to the same kind of analysis. This is particularly true of each proof, should there be any in the text. Is the overall strategy of the proof evident (or is it explained) and is the argument simple and clear? Does it relate apparently disparate concepts in a surprising and insightful manner or does it force different ideas together in an artificial and confusing way? With regard to the definitions, are they formulated in a clear and pleasing fashion and placed in the text in a way that enhances the structural cohesiveness and promotes the perspicuous flow of the argument?

Finally, we should consider the author's particular style. Is his/her writing, including the use of such resources as tables, graphs and diagrams, aesthetically pleasing and does it contribute to the reader's understanding of the text? Is the use of mathematical symbolism

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appropriate and perspicacious? Does the text reveal the author's excitement about and/or engagement with the ideas that are being presented? Does it engage the reader and pique his/her interest, or does it become an obstacle that the reader has to overcome in order to understand the text?

EXTERNAL BEAUTY

External beauty is like that of its internal counterpart, except in that the text now plays the role of a subpart of a greater whole. That is, whereas internal beauty is concerned with the harmonious relations of the various components of the text being evaluated, external beauty is concerned with the harmonious relations of the text itself with other parts of any greater wholes to which it may belong.

The immediate question is that of how the text relates to the mathematical tradition. Does it, for example, complete, extend or generalize earlier results in a fitting way? Does it simplify existing theories and/or resolve any outstanding *ad hoc* assumptions? One profitable way of approaching external beauty is that of asking whether the text increases the internal beauty of any greater whole to which it belongs.

Clearly, mathematics, as a cultural phenomenon, is related to other, non-mathematical aspects of human culture and, thus, we may extend our investigation of the external beauty of a text to considerations about the harmonious relations that it may maintain with extra-mathematical aspects of our culture. Again, the basic question to ask is whether the text contributes to the internal beauty of culture as a whole. Does it, that is, participate in making human life more fulfilling and more meaningful? Answering such considerations may require a breath of knowledge beyond that which could reasonably be expected from the student. Nevertheless, it would seem appropriate to introduce the student to wholistic modes of thought even at an early age, so that s(he) may begin to develop an appreciation of the integrated nature of human culture in general and a greater awareness, specifically, of the role that mathematics plays in that culture.

INTERNAL GOODNESS

A text has internal goodness if it is good in and of itself, that is, if it is a contribution to the advancement of mathematical knowledge. Perhaps the most evident way of making such a contribution is by way of proving new theorems. This is, albeit, not the only way that mathematics can be advanced. Making conjectures, generalizing, clarifying and simplifying, as well as giving counterexamples or otherwise disproving or reformulating mathematical propositions all are valuable contributions to the mathematical enterprise. Further, although possibly less evidently, texts may make important contributions by applying or otherwise relating mathematics to other cultural phenomena or by reflecting on the history and philosophy of mathematics.

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EXTERNAL GOODNESS

In contrast to the internal development of mathematics posited by internal goodness, its external counterpart promotes the advancement of mathematical knowledge, that is, it contemplates contributions that the text makes to the advancement of extra-mathematical knowledge by mathematical means, either on the micro or macro level.

Contributions on the micro level are those relating to the intellectual development of the reader. In this regard, our judgement should be tempered by considerations relating to both the author's intended audience and the historical situation at the time of the text´s publication. A research article on the frontiers of mathematics, for example, may seem to have but little, if any, effect on the intellectual development of a non-mathematician and yet be of immense value for a mathematician researching the same or a related area. Texts also have a history and may become more or less pertinent and contributory as the field advances or retreats.

On the macro level, external goodness refers to the advancement of extra-mathematical knowledge by mathematical means. The most obvious way of doing so is by applying mathematics to other cultural areas. Observe that we have already seen, in the previous section of the present work, that applied mathematics may advance mathematical knowledge itself. It may also (in fact it generally does, otherwise the application would not be made) advance extra-mathematical knowledge. In such a case, the text would have high levels of both internal and external goodness.

In any case, it is interesting to draw out a consequence of the difference between internal and external goodness. The former *is itself* a contribution to mathematical knowledge, whereas the latter *promotes* mathematical knowledge and its relations to other parts of the culture. This would seem to imply that the latter category may include texts that are not strictly speaking mathematical in nature. These may range from textbooks[8] (generally thought of as mathematical texts), to reflections on the nature of mathematics, to texts on Mathematics Education (not generally thought of as mathematical texts). Recall that I prefaced the discussion of my hermeneutical structure by pointing out that, in the present paper, I would consider mathematics as a primitive term. I did so precisely in order to allow the reader to adopt as inclusive or exclusive a stance as s(he) feels comfortable with in regard to this term. Those who take a more exclusive posture will have to make slight adjustments to my remarks regarding internal and external goodness. Nevertheless, as long as they are willing to make the reasonable stipulation that their non-mathematical texts may promote the advancement of mathematical knowledge, their adjustments will be generally consonant with my analysis.

A TABULAR SYNTHESIS

The foregoing analysis can be synthesized by isolating the leading characteristics of each of the eight categories and presenting them in the following compact form:

[8] Cf., Fossa (2021).

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	Internal	External
Unity	Independent, self-contained presentation Free of extraneous content	Position in the flux of mathematical culture Extra-mathematical relations
Truth	Internal consistency Adequacy of proofs	Adequacy of coherence with other unities in which it participates
Beauty	Structural coherence Nicety of formulations	Fit with other math theories Fit with extra-mathematical concerns
Good	Contribution to mathematical knowledge	Promotion of mathematical development in the individual and/or the general culture

As a synthesis, of course, the structure is deprived of the details that gives it a robust description of the many hermeneutical possibilities of the reader's interaction with a mathematical text. Nevertheless, it may serve the teacher well in his/her initial planning of lessons based on the reading of original texts, including establishing the objectives of the lesson. As the planning evolves, the teacher can go back to the details that s(he) may need to consult.

TWO GLOBAL STRATEGIES

It has not been my intention herein to describe the implementation of pedagogical strategies that would help the student to interpret the text in fruitful ways with regard to the kinds of interactions categorized in the proposed hermeneutical structure. In general, such strategies revolve about guided readings, involving the posing of

(*i*.) questions that direct the student's attention to relevant aspects of the text

(*ii*.) debates that challenge certain interpretations and require the student to inquire more deeply into the meaning of the text

(*iii*.) tasks, such as filling in lacunae in proofs, that ask the student to go beyond what is immediately given in the text.

There are various examples to be found in the literature on this topic, such as the description of the History of Mathematics Project given in Kjeldsen and Blomhøj (2009) and the discussion provided in Barnett, Lodder and Pengelley (2014).

Since most of these descriptions tend to deal with more localized aspects of the text, however, I will briefly describe two strategies that may help the student to come to terms with some of the larger structural issues that come into play with regard to, for example, internal beauty.

The first of these is akin to the technique of skimming through a literary text by reading only the lead sentence of each paragraph. Analogously, with regard to mathematical texts, it is often instructive to read the sequence of theorems presented in the text, without paying attention to the demonstrations and other intervening material. In this way, the deductive organization of

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the whole text becomes apparent so that we can determine whether it tells an interesting and compelling story and whether it does so in a pleasing and satisfying manner.

The other technique for determining global issues is to trace the deductive dependence of the theorems on each other. One way of doing this is to make a simple tree diagram of this dependence, such as the ones used in Fossa (2014) to exhibit the deductive structure of Frenicle's *Traité des Triangles rectangles en Nombres* (1676).

Both of these techniques are facilitated when the text investigated is one that is organized on a theorem-demonstration basis. Should that not be the case, it will be incumbent on the reader to ferret out the theorems, or perhaps the theorem-like propositions, contained in the text. Of course, these techniques will not be applicable to all historical mathematics texts, but when they are, they can be quite useful in determining its overall structure.

CONCLUSION

One of the great advantages of using the history of mathematics as a pedagogical resource is that it puts the student in the role of a researcher in mathematics, not, of course, at the frontiers of mathematical knowledge, but at the frontiers of the student's own knowledge. The reading of historical texts in the classroom, as the hermeneutical substructure herein described plainly indicates, takes a further step in that it can place the student in the role of the historian of mathematics. It can also introduce the student to a welter of issues and concerns that go beyond the truth claims made in the text. By considering also the unity, beauty and goodness of the text, the student is introduced to wholistic modes of evaluation that give him/her a greater appreciation of what mathematics is and how it is related to the many faceted wonderland of human culture. Thus, the hermeneutical interaction with original texts is a powerful tool in accomplishing one of education's – including Mathematics Education's – fundamental tasks, that of helping the student to appropriate human culture and thereby enrich her/his life.

Recall that the substructure is not intended to be a template to be strictly implemented in all contexts of reading historical texts. Rather, it is a resource from which aspects may be taken to help organize and direct the student's interactions with the text. In this regard, given that questions about the truth of the text are those most familiar to both the student and the teacher, an interesting way of implementing the substructure would be to consider truth and unity with regard to one text, truth and beauty with regard to another and truth and goodness with regard to a third. Each teacher will also have to determine just what is to be expected of the student in each teaching situation and the level of external support – which may be accomplished by team-teaching with colleagues from other disciplines – to be given to the student. As is almost always the case in education, however, any investment by the teacher will redound to greater benefits for the student.

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