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Teach plane geometry in coherence from 6 to 15 years

Ensinar geometria plana de forma consistente dos 6 aos 15 anos

Enseñar geometría plana en consistencia de 6 a 15 años

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ABSTRACT

This article is based on an unpublished text available on HAL (https://hal.science/hal-01660837v2). It is a reflection on the teaching of plane geometry in the French context, in order to envisage a coherent approach throughout compulsory education, taking into account the knowledge usually ignored in teaching. After an analysis of the difficulties in teaching geometry, we draw on Duval's work and Brousseau's theory of situations to propose an approach to geometry from the analysis, reproduction and construction of figures with tracing instruments, excluding measuring instruments, by explaining the rules of geometric use of these instruments. The aim is to conceptualize the basic theoretical geometric objects and their relationships, including straight lines, circles, points, angles.
Keywords: Plane geometry; Compulsory education; Analysis of geometric figures; Plotting instruments; Graphic space.

RESUMO

Este artigo é um resumo e um suplemento de um texto não publicado disponível em HAL (https://hal.science/hal-01660837v2). Trata-se de uma reflexão sobre o ensino da geometria plana no contexto francês, com o objetivo de desenvolver uma abordagem coerente ao longo da escolaridade obrigatória que tenha em conta as aprendizagens habitualmente ignoradas no ensino. Depois de analisar as dificuldades do ensino da geometria, apoiamo-nos nos trabalhos de Duval e na teoria das situações de Brousseau para propor uma abordagem da geometria baseada na análise, na reprodução e na construção de figuras com instrumentos de desenho, excluindo os instrumentos de medida, e na explicação das regras de utilização geométrica desses instrumentos, com o objetivo de concetualizar os objectos geométricos teóricos de base e as suas relações, nomeadamente rectas, circunferências, pontos e ângulos.
Palavras-chave: Géométrie plane; Scolarité obligatoire; Analyse de figures géométriques; Instruments de tracé; Espace graphique.

RESUMEN

Este artículo es una versión sintética y completa de un texto no publicado disponible en HAL (https://hal.science/hal-01660837v2). Se trata de una reflexión sobre la enseñanza de la geometría plana en el contexto francés, con el fin de concebir un enfoque coherente a lo largo de la escolaridad obligatoria que tenga en cuenta los aprendizajes generalmente ignorados de la enseñanza. Después de un análisis de las dificultades en la enseñanza de la geometría, nos basamos en el trabajo de Duval y en la teoría de las situaciones de Brousseau para proponer un enfoque de la geometría a partir del análisis, de la reproducción y construcción de figuras con instrumentos de trazado, con exclusión de los instrumentos de medida, explicitando reglas de uso geométrico de estos instrumentos, destinadas a conceptualizar los objetos geométricos teóricos básicos y sus relaciones, especialmente rectas, círculos, puntos, ángulos.
Palabras clave: Geometría plana; Escolaridad obligatoria; Análisis de figuras geométricas; Instrumentos de trazado; Espacio gráfico.

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INTRODUCTION

The study of geometric figures is important in the culture of the citizen in order to deal with the problems of space in everyday life and also for certain professions, and therefore for certain branches of professional schools. It makes it possible to lay the foundations of a geometric vision of the world and to create the intuition that goes with it and that will serve to solve problems in other areas. Geometric quantities and their operations are an important support for the extension of integers and their operations to decimals, rational and real numbers, and even beyond, for algebra. Length transfers are the basis for building the graphical register as a frame at the interface between quantities and numbers. Even the treatment of geometry through computation, is made more effective with the support of geometric knowledge in relation to numbers.

But what is meant by the study of geometric figures and how should it be taught if we want to think about it consistently throughout compulsory schooling? Much research has pointed to the rupture at the beginning of secondary education between a geometry of material figures (on paper or on the screen) with instruments and a theoretical geometry of statements and demonstrations. Houdement and Kuzniak (2006) even speak of two paradigms. This rupture is due in particular to the way in which statements are validated, in the first case by the perception, with the aid of instruments, in the second case by a logical discourse of statements describing properties of theoretical objects or relations between these objects. However, this research rarely considers the teaching in all compulsory schooling. The purpose of this article is to reflect on a progressive construction of a theoretical point of view, by making play a positive role in the access to theoretical geometric objects to the validation through the instruments of geometry (rule, square, compass), by encouraging the search for general properties and the continuous enrichment of theoretical knowledge.

GOALS AND DIFFICULTIES IN TEACHING GEOMETRY

Geometry as a model of space and theoretical geometries

The teaching of geometry in compulsory education has several objectives (KAHANE, 2002), including the acquisition of knowledge in order to model concrete problems that arise in the sensitive space to solve them, the development of mathematical reasoning, that is, a means based on logic, proof, to validate the solution of problems that are not easily made algorithmic, and finally, the development of geometric thinking or geometric intuition, which is a powerful heuristic tool for transferring intuitions from our relationship with space to other areas of knowledge, including within mathematics. These goals correspond to two aspects of geometry: geometry as a model of space with a practical purpose and geometry as a coherent mathematical theory based on an axiomatic foundation. In a reworking of an ancient text, Brousseau (2000) imagines two fundamental situations in order to distinguish them: The situation of the carpenter who cuts heavy pieces of wood on the ground and has to adjust them precisely when he assembles them ten meters from the ground and the situation of the intersection of the perpendicular bisectors of the sides of a triangle, which consists in drawing on paper, with the usual instruments, a triangle with a very obtuse angle and its three perpendicular bisectors. Because of the imprecision of the drawings, they usually form a very small triangle; it is then a question of finding another triangle for which the triangle formed by the three bisectors is as large as possible. The situation of the carpenter represents a part of the useful knowledge to deal with problems that arise in the sensitive space, which fall under what we will call a physical theory of space corresponding more or less to the paradigm GI of Houdement and Kuzniak (2006). In the case of the bisectors, the search for such a triangle leads to the proof that it cannot exist: to ensure the logical coherence of the model, the three bisectors must intersect at the same point. This situation represents for us the elementary theoretical geometry corresponding to the GII paradigm of Houdement and Kuzniak (2006), taught in secondary school.

Several choices of axiomatics are possible and have been made at different times to found the teaching of geometry in secondary school in France, including the axiomatic of Euclid in different versions, an axiomatic based on transformations and in particular orthogonal symmetry (COUSIN-FAUCONNET, 1995) and of course the notion of Euclidean affine space taught at the university. Whatever the theory, it can be used to solve problems that arise in the sensitive space because the axioms have not been chosen at random. However, the underlying axiomatic is not indifferent to think the coherence and the continuity of the teaching of geometry through all compulsory education. In fact, by the age of 6, it is clear that we cannot use anything other than direct perception to describe or reproduce the shapes, positions, or movements of objects in space. For example, we learn to recognize a rectangle by relying on visual and tactile perception (which is important to develop at this level). Between the ages of 6 and 12, the child is gradually introduced to geometric tools that help him or her perceive in a more accurate and secure way. For example, a rectangle can be justified by checking the right angles with a square and the lengths with a graduated ruler. If we want to ensure a certain coherence, the teaching of geometry in secondary school should be based on this work with the tools to build a theoretical model that allows to be sure without realizing the experiment, using already known properties. But many of these properties are admitted with the students, how then to be sure of the coherence of the model? Teachers in charge of the introduction of this theoretical model and of the mathematical proof should have an underlying axiomatic to feel legitimate and assured of the validity of what they teach. However, the model of the Euclidean affine space does not allow to found a teaching based on the sensitive space of this type, because it requires to take as prime objects the points and vectors that do not come from the direct perception of the space. This model, however effective it may be for the continuation of studies, cannot help teachers who have to manage the conflicting relationships of their students between perception in the sensitive space and reasoning in the theoretical model of this space. It can only come a second time if we already have a first model of geometry based on the sensitive space. This is what Gonseth said:

> Certes le géomètre peut refuser de comparer son expérience avec celle du physicien. Il peut s'enfermer dans des systèmes axiomatiques, en posant comme données a priori les axiomes et la logique de la déduction. Il évitera ainsi, de justesse, le problème de l'espace (…) mais il n'aura rien fait pour l'éclairer. » GONSETH, 1945, I-8.
>
> (Certainly, the geometer may refuse to compare his experience with that of the physicist. He may enclose himself in axiomatic systems, laying down the axioms and the

> logic of deduction as data a priori. In this way he will narrowly avoid the problem of space (...), but he will do nothing to shed light on it.)

We will return to this choice of underlying axioms. Let us first consider the role of material figures, drawn on paper or on a computer screen, in the practice of geometry.

Graphic space. The dual role of material figures

At the heart of the problem of teaching geometry is the relationship between the geometric objects and the material objects of sensible space, the theoretical figures defined by relationships between geometric objects and the material figures[2] that represent them, controlled by perception or instruments.

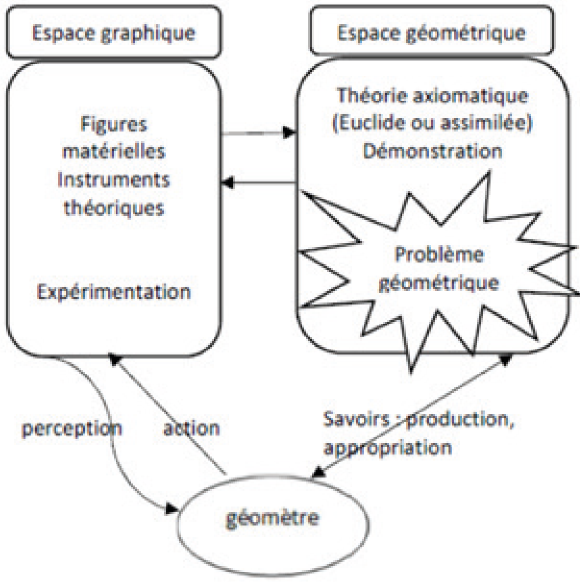
Representations, diagrams, and figures play different roles depending on whether the problem to be solved occurs in the real world or in a theoretical model. When the problem occurs in the real world, drawings and models can represent objects in the real world. These drawings or models, in turn, can be modeled by figures representing geometric objects, which can lead to draw them again. Diagrams and figures thus act as intermediaries between the sensitive world and theoretical geometric objects. To use the results established in a theoretical model of geometry, it is necessary to identify in the concrete problem the elements that can be translated into the theoretical model both as assumptions and as conclusions of the result that is used. Representation is one way to do this. Therefore, in addition to the sensitive space, we consider a specific part of the sensitive space which we call the graphic space of representations. There are representations of objects of the sensitive world or geometric objects in the form of diagrams or figures in dimension 2 (D2). The graphic space is an interface between the sensitive space and the geometric space in the case of a problem posed in the sensitive world, a case that we will not consider in the rest of the text. For more details, see Perrin-Glorian et al. (2013), or Perrin-Glorian & Godin (2018), where we analyze a problem from a professional high school textbook calculating the area of the glass surface of the Louvre pyramid. If the problem is theoretical, it concerns theoretical objects that can be represented in the sensitive space by models, figures, whether they are made with classical instruments or with software. The graphic space (or even the sensitive space for models) then serves as an experimental ground for this theoretical problem.

In fact, the study of figures in compulsory schooling usually has neither a practical nor a theoretical purpose, but rather a conceptual learning purpose. Depending on whether the object of study is the material figure or the geometric figure, we have two radically different ways of validation, corresponding to what we have called the physical theory and the axiomatic theory of geometry.

2 Much research on the teaching of geometry distinguishes between drawing and figure, and this distinction is commonly used in teacher training. However, geometric drawings are not just any drawings, but drawings with characteristics that can be produced with well-defined geometric instruments (in fine the ruler and the compass). We prefer the term "material figure" to "drawing" to emphasize that it is these characteristics that we are interested in. Moreover, it seems important to us that the terms "reproduction of figures" and "construction of figures" can refer to both material and theoretical figures (in the case of construction).

When the object of study is the geometric figure, we have the diagram of Figure 1:

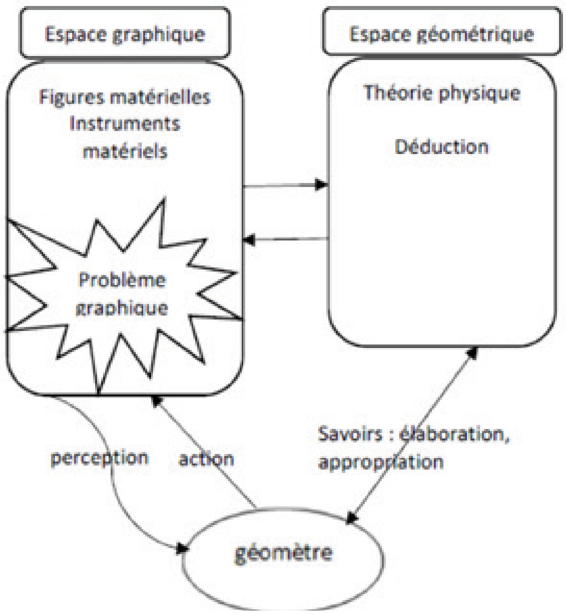
Figure 1 – Problem in the geometric space.



The material figure is a representation of the theoretical geometric figure, and the reference theory is an axiomatic theory (even if the axioms are not explained to the students). The validation of propositions is done by deduction from statements. Some statements are called axioms; statements that have already been validated and deserve to be preserved because they are often used are called theorems. Other statements are called hypotheses: they are specific to the figure studied.

When the object of study is the material figure, in elementary school or even at the beginning of secondary school, we have a graphic problem that corresponds to the diagram in Figure 2: the reference theory is a physical theory. In the following we will specify how this physical theory can be built within the framework of what we call "the geometry of tracing".

Figure 2 – Problem in the graphic space.



In Perrin-Glorian and Godin (2018), we find the example of the same problem treated in both cases: construct a right triangle knowing (in magnitude) a side c of the right angle and the hypotenuse a, where one of the two is also known in position (i.e., given by a segment on which to construct).

The study of material geometric figures: continuity and rupture between primary and secondary school

Continuing Brousseau's work, Berthelot and Salin (1993, 1998) distinguish spatial knowledge, which is used to manage our relationships with space, and geometric knowledge, and are interested in the relationship between spatial knowledge and geometric knowledge by distinguishing two ways of dealing with the problems posed in sensitive space. They identify three problematics in the relationship between sensitive space and geometry:

- The *practical problematic*: the problem is posed in the sensitive space, the relations to the space are effective, they are controlled empirically and contingently by the senses, the validation is done in the sensitive space.
- The *modeling or spatio-geometric problematic*: the problem is posed in the sensitive space, but it cannot be treated directly in this space: it is translated into a geometric model where the resolution is made; the result is translated back into the sensitive space and the validation is also done in the sensitive space.
- The *geometric problematic*: the problem, the treatment and the validation are done within the framework of theoretical geometry, according to established rules. Relationships to space, for example to the figure, can be effective, but are governed by the definitions and rules of operation of the theoretical objects it represents.

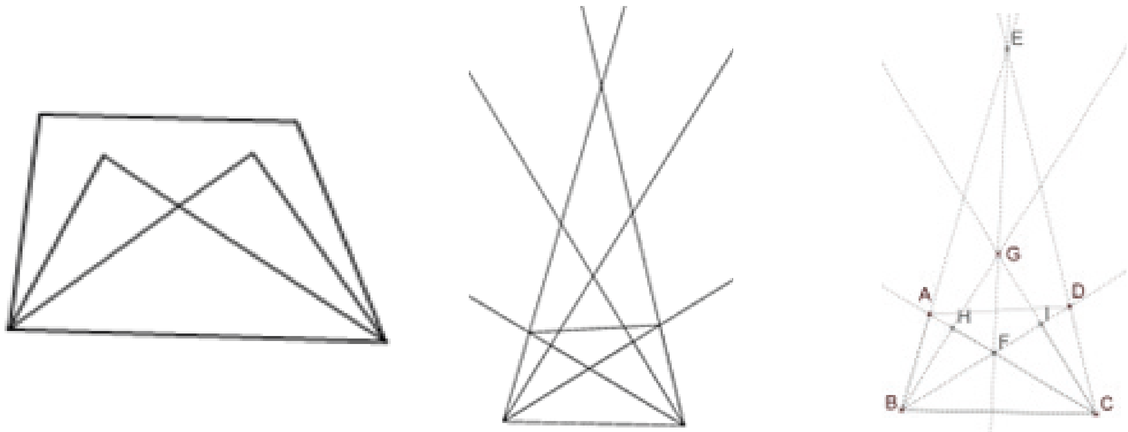
Berthelot and Salin (1993, 1998) consider that, for students, the practical problematic hinders the geometric problematic, especially in the micro-space of the sheet of paper. They show that the size of the space is an important didactic variable, because the control by sight and by material instruments does not have the same efficiency according to the size of the space. Thus, 50% of the students at the end of primary school are unable to predict the position of the feet of a rectangular bench after a displacement, whereas most of them are able to draw a rectangle on a sheet of paper where they can use the visual control continuously. They then propose the spatio-geometric problematic as a way to enter the geometric one from problems posed in space, but with blocking of natural spatial procedures, especially by working in the meso-space. They thus retain three classes of variables important for the analysis of situations of geometry in education: the type of problematic, the type of relation to space, effective or internalized, the adidactic character or not of situations.

For our part, we are interested in the work on figures on paper, that is, in the micro-space, which we have called graphic space. In theoretical geometry, corresponding to GII or the geometric problematic of Berthelot and Salin, the object of study is a theoretical figure defined by properties (given by statements or coding). Validation is done by deduction using already validated properties (see Figure 1). When the material figure is the object of the work, the validation is done by tracing instruments that control graphic characteristics corresponding to the representation of geometric properties in GII (Figure 2). There is therefore a break in the relationship between these two modes of validation. However, there is continuity in the statements between GI and GII: the valid properties are the same[3] and the material figures are the same.

To acquire a mobility of view on the material figures

Proof in geometry requires the articulation of the graphic register of figures and the verbal register of language. Duval's work (1998, 2005) has shed light on the cognitive difficulties involved in articulating these two registers. Indeed, geometric activities involve three cognitive processes that perform different epistemological functions: visualization, construction, and reasoning, which can be performed separately but must be linked in geometry (DUVAL, 1998, p.38). Natural visualization favors figural units of dimension 2, while the figural units invoked in geometric statements are most often of dimension 1 (lines) or 0 (points). Reasoning about figures therefore requires a dimensional deconstruction of these figures, and their construction also requires considering the constraints of the instruments. Building on Duval's work and considering the different ways of describing and constructing plane figures, we have distinguished three main conceptions of composed[4] plane figures, taking into account the dimension of the figural units, but also the ability to see untraced figural units useful for constructing or defining elements of the figure. This distinction takes into account not only the visualization of figural units, but also the determination of figural units by others. The "surfaces" conception is to see the figure as a set of juxtaposed surfaces or, less naturally, overlapping surfaces. In a "lines" conception, the figure is defined by a network of lines that can be drawn with instruments: the ruler for the lines, the half-lines (that can be extended) and the segments, the compass for the circles or arcs. In a "points" conception, the figure is defined by points. We think of these conceptions as nested: if you have a points conception, you also have a lines conception and a surfaces conception.

Figure 3 – Three conceptions of the same figure.



3 Not quite in the case of the approached constructions that we leave aside here.

4 We call a simple figure a figure that can be obtained from the outline of a template, such as a nonintersecting polygon, and a composed figure any other plane figure. A simple figure can be thought of as a surface with an edge that does not intersect, or as a closed line that does not intersect and thus encloses a connex part of the plane, and a composed figure can be thought of as a set of simple juxtaposed or superimposed figures, or as a set of lines, some of which are closed and some of which are open.

For example, in the figure to the left of Figure 3, using the surfaces conception, you can see three adjacent triangles or two overlapping triangles, all on a quadrilateral. In a surfaces conception, lines and points can appear, but lines are only edges of surfaces, and points are vertices of surfaces or, in the case of overlap, intersections of edges. You can't create new lines without moving a surface. In a lines conception, you see the figure (in the center of Figure 3) drawn on a network of lines (here, half-lines). The lines coming out of the convex envelope of the original figure remain difficult to conceive. Points are ends of lines or intersections of lines that you already have. You can draw segments (or even lines) connecting points you already have. In a points conception of the figure, you can create new lines to get a point at their intersection, and the lines are defined by points. You can thus see the figure (right in Figure 3) as defined by points A, B, C, D, G (E and F are obtained by intersecting lines that can be traced from A, B, C, D). We name the points here to facilitate communication, but it is not necessary if you analyze the figure without communicating the result.

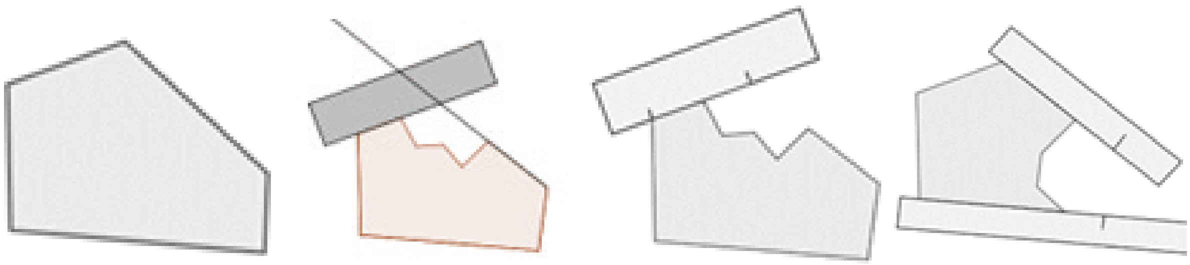
Note that even with the points conception, lines are not necessarily seen as sets of points.

REPRODUCTION OF GEOMETRIC FIGURES AS A SOURCE OF PROBLEMS

What does reproducing a geometric figure mean?

To reproduce a figure is a common task in elementary school. It can take on different meanings depending on the criteria used to judge the fidelity of the reproduction, and the instruments available to carry out this reproduction. From kindergarten, children learn to reproduce figures in the form of a puzzle with geometric shapes, flat objects in space that are manipulated. They then use only perception, visual or tactile (for example, to recognize a straight edge). They will soon be able to reproduce a figure by drawing the outline of a template. If a corner is missing from the template (Figure 4), it will be necessary to use one straight edge of the template or a ruler to extend two sides to get an intersection, or to extend one side and transfer a length. If an entire side is missing, the length transfer will be necessary. Thus, we can act very early on the tools available to the students to change the nature of the problem we pose to them and promote the development of their knowledge.

Figure 4 – To reproduce a polygon with a torn template, a ruler, and length transfers.



In the previous example, we were interested in reproducing a polygon identically: the shape and size of the polygon are preserved, we can verify the accuracy of the reproduction by superimposing a tracing paper. In general, for the study or reproduction of a material figure drawn on paper with a material validation method, two cases should be considered:

either the figure is reproduced identically and the equality of the figures can be checked by superimposing them with a tracing paper; or the reproduction is made in a different size and the adequacy with the model requires the use of geometric properties such as the conservation of angles, alignments, length ratios. It can also be done by checking the properties of the figure to be reproduced, for example an equilateral triangle or a square (the number of sides determines a regular polygon, modulo similarity, in other words, its name determines its shape). Note that this is not the case for a non-square rectangle or a non-equilateral isosceles triangle: for an isosceles triangle you must also respect the angles or a length ratio, for a rectangle you must respect both the angles and a length ratio. The triangle plays an important role in the reproduction of figures in different size, since the preservation of the angles ensures the preservation of the shape: triangles with equal angles are similar, which is not the case with quadrilaterals. If the figure is reproduced identically, the shape and magnitudes are preserved; if the figure is reproduced at a different size, the shape is preserved, also the angles, but not the lengths, only the ratios between the lengths are preserved. In geometry, preserving the shape means producing a figure similar to the model, but the word "geometric shape" has a much broader meaning, even in the course of geometry in earlier years.

In kindergarten, material objects come first: children discover and formulate properties that allow them to recognize them through visual perception or touch. These properties correspond to the visual characteristics of the objects themselves or to the graphic traces that these objects leave on the paper when the outline is drawn. You can create graphic traces of material objects without drawing the outline, with tools that reproduce and verify the visual characteristics of these traces. These visual characteristics correspond to the properties (length equalities, right angles, parallelism) that allow the geometric objects to be defined. The vocabulary introduced on material objects remains the same, or almost the same, when it is applied to graphic traces, then to the geometric objects that these traces represent. During compulsory schooling, therefore, there are two major turning points in the way figures are defined: from mere perception to the control of properties by means of instruments, then from control by means of instruments to control by means of statements.

The usual instruments, the ruler, the square, the compass, play an essential role in the transition from the control of the figures by the sole perception to the control by the statements, because of the instrumental deconstruction that they require. (MITHALAL; BALACHEFF, 2019). In addition, the common tools generally perform several functions that we find necessary to distinguish in learning because they are related to the conceptualization of different geometric notions. For example, the ruler has three functions: to create or check alignment, to transfer a length, and to measure a length. In the next section, we explain why we exclude measurement from our approach before looking at other functions of tracing tools.

Magnitudes and measures

Geometric magnitudes, namely comparisons and transfers of lengths and areas, play an essential role in the conceptualization of integers and rational numbers and their operations, but also of square roots, and even limits and thus other irrational numbers through the intuition of the continuous they allow. In the perspective of a coherent progression of geometry coordinated with a progression of numbers throughout the compulsory schooling, it is essential to approach the operations (addition, subtraction, multiplication, and division by an integer) on geometric quantities (first lengths, then angles and areas) independently of their measurement, so that they can serve as a support for the construction of numbers. Of course, we are not saying that it is not necessary to learn to measure with a graduated ruler or a protractor. But for us, measure is on the side of numbers, and in order for it to play its role in the conceptualization of numbers, it is also necessary (first or simultaneously) to work on the end-to-end setting of lengths, the addition of areas and angles.

Thus, we are interested in the instruments of tracing and the transfer of lengths or angles and not in the instruments of measurement in what we will call the geometry of tracing. The geometric magnitudes are present, as well as the transfer of magnitudes and the ratio of magnitudes but not their measurement: if necessary, the size of the figures is fixed by lengths given by segments and not by their measurement.

Tracing instruments, interface between the perceptual and the theoretical

To think about the continuity of the teaching of geometry in compulsory education, we include templates, stencils and tracing paper among the material tools used to trace and reproduce figures (simple or composed) on paper. We will therefore consider different tracing tools that allow the transfer of shapes (parts of surfaces or lines and relations between lines) and magnitudes (angles or lengths). These tools require a more or less elaborate vision of the figure and the use of more or less geometric properties. Templates and stencils make it possible to convey all the information on simple figures. Tracing paper, if large enough, allows all the information on the figure to be reproduced, whether simple or composed. Other tools make it possible to carry information of dimension 2 (D2) on the figure without carrying all the information: template or torn stencil, too small tracing paper, piece of opaque paper on which one can write, but also square, compass, parallel ruler. The square, for example, carries information on a piece of surface between two straight edges. On the other hand, the ruler only allows you to check and report alignments, i.e., information of dimension 1 (D1). The compass makes it possible to reproduce circles; by doing so, it carries information D2 (a disk or part of a disk), but to do so, it is necessary to identify two points: the center of the circle and a point of the circumference, or a point and a magnitude. Reasoned use of the compass (with justification) therefore requires a points conception of the figure: to know that whatever point you take on the circle, it will be at the same distance from the center. However, even with access to the points conception in some situations, even if they can identify certain points on lines, most students at the beginning of secondary school do not see circles as sets of points, any more than they see straight lines or segments.

GEOMETRY OF TRACING

We will now define a certain relationship to elementary geometry, which consists in the study of figures, and which we have called "geometry of tracing" because it does not involve measurement. The fundamental situation of this geometry is the reproduction or construction of figures with instruments. The reproduction of figures requires going from figure to figure, from text to figure. In order to establish a geometric relationship to the figures, the geometry of tracing requires explicit rules for the use of geometric instruments and the establishment of a language to describe the figures and their construction.

Theoretical instruments and material instruments

In theoretical geometry, material figures are drawn freehand or with instruments, they support reasoning; the working figure is the geometric figure (immaterial). In physical geometry, material figures are drawn with concrete instruments that have a limited field of efficiency; freehand drawings, possibly coded, are diagrams that help memory and are to be realized with instruments. However, the functions of the usual geometric instruments are linked to geometric concepts. The ruler (not graduated) allows you to draw straight lines; it is linked to the notion of alignment. The square allows to draw lines and to transfer right angles; it is linked to the conceptualization of the right angle with its two infinite sides. The compass makes it possible to draw circles and also (later) to transfer lengths on a line already drawn (by creating the intersection of a half-line and a circle). We consider theoretical instruments, which do not have the limitations of material instruments, to represent the different functions of material instruments, while considering how they can be realized materially. For figures used in proofs at the beginning of secondary education, the only theoretical instruments we will need are *the ruler* and *the compass*, but when knowledge is limited, other instruments are needed to transfer geometric magnitudes without resorting to measurement and numbers.

- The *ruler* (not graduated) allows you to draw straight lines as big as you want. It represents the straight line.
- The *length-transfer* allows you to transfer lengths on a line that has already been drawn. It can be made by a strip of cardboard that cannot be folded, with a straight edge on which you can write. The dry point compass (with two identical branches) has the same use, but the length to be transferred is not materialized by a segment.
- The *length-bisector*[5] allows you to take half a length. It can be made by a foldable paper strip with a straight edge, on which you can write.
- The *angle-transfer* allows you to transfer angles from a line and a point on that line. This can be done using a template, a folded piece of paper or a circle drawn on transparent paper with one of its radii. A point is marked as the vertex of the angle.
- The usual *compass* allows you to draw circles.

Rules for geometric use of instruments

For the material instruments to play the role of interface between physical geometry and theoretical geometry, it is important that the students learn to use them in a geometric way, which we call geometric use because it respects rules corresponding to the representa-

5 We introduce this word to facilitate the writing of this text. We do not propose to introduce it to the students because it is a temporary tool in the geometry course. Later, they will have the knowledge to find the center of a segment with a ruler and a compass using the perpendicular bisector.

tion of geometric properties by the theoretical instruments and implicitly refers to axioms, definitions, or geometric theorems.

- To set up the *ruler*, you need two points, or an already drawn segment.
- To set up the *length-transfer*, you need an already drawn line and a point on this line from which to transfer the length.
- The *length-bisector* allows you to take the length of a segment and to divide it into two parts; to take the center of a segment, you transfer the length of this segment to the length bisector, then you transfer the half-length to the segment from one end.
- To set up the *angle-transfer* (for example a template), you need a straight line and a point on this line: the top of the template is placed on the point and one side of the template is placed on the line, the angle is drawn on one side or the other. To draw parallel straight lines, drag one side of an angle template onto a straight line and draw on the other side.
- The compass has two distinct branches: the point and the mine. To draw a circle with a given center passing through a given point, place the compass point on the center and the mine on the point; it describes a circular arc when you turn this branch around the other. To reproduce a circle, it is necessary to locate the center and open to a point of the edge. This distance represents a length: the distance between the center and any point on the circumference. A length can therefore be transferred from a point on a line with the compass by placing the point on that point and drawing an arc that intersects the line.

This learning is not currently supported by teaching. However, we believe it is essential to clarify the connections between physical and theoretical geometry, as it emphasizes the accuracy of construction rather than its precision (PETITFOUR, 2017). In effect, the use of instruments must be related to the purpose of geometry: if the purpose is practical, one seeks to have as much precision as possible according to the instruments available, or to adapt the use of instruments to obtain more precision in the trace, what we call a technical use of instruments; if the purpose is theoretical, one seeks to have a correct construction thanks to a geometric use of instruments. A challenge of teaching at the end of primary school or at the beginning of secondary school is to distinguish these two uses.

Students can use the compass as a length transfer when they see the distance between the point and the mine as a length and the circle as a set of points at the same distance from the center. As students' knowledge develops, they have a repertoire of theorems and construction processes that allow them to do without the length bisector and the square and then the angle transfer. The related techniques can meet earlier in the realization of particular figures. For example, the center of a segment can be built in many ways (PERRIN-GLORIAN; GODIN, 2018, p. 23-24).

Restoration of geometric figures as an action situation

We have said that the reproduction of figures is the fundamental situation of the geometry of tracing, insofar as it allows to generate all the situations for teaching this geometry. Among the reproductions of figures, we were particularly interested in a type of situation that we called the restoration of figures. We developed and studied it in order to work with students from 6 to 12 years of age on the mobility of the view of geometric figures (in particular, the transition from the surfaces conception to the lines and points conception), as well as the geometric use of material instruments. We define it from the characteristics of a situation in the theory of situations: the problem and the rule of the game, the milieu and its didactic variables, the knowledge at stake.

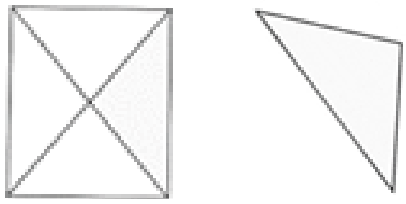
Problem and rule of the game : A figure restoration is a reproduction of a material figure, but with particular constraints: a model figure is given (in real size or not); a part of the figure to be obtained (starting figure) is given either by its drawing, either by an instrument that makes it possible to transfer D2 information from the initial figure, but without giving all the information (for example, a part of a template); we have different instruments to which we can give a cost of use in a scale; when the students think they are ready, they can test their production with a tracing paper (with the solution) available from the master.

The milieu consists mainly of the model figure, the starting figure, the instruments with their costs. The didactic variables refer to the choice of these different elements of the milieu.

The knowledge involved must be examined in each case. The milieu and the rules of the game are chosen according to the knowledge that is supposedly available and that we want to encourage to emerge.

It is a matter of restoring material figures with material instruments. The model figure provides information about graphic characteristics that can be identified and reproduced with the instruments. However, some of these graphic characteristics represent geometric properties that can be recognized as such and from which we can derive new geometric properties that translate into new graphic characteristics. For example, in a quadrilateral, two opposite vertices are aligned with the intersection of the diagonals. The choice of the didactic variables is made in such a way that the use of the geometric properties that one wants to work with allows to reduce the cost of the restoration. Thus, the goal of figure restoration is to encourage the identification of lines and points that allow the figure to be geometrically defined and constructed. If the goal is to help students move from a surfaces conception to a lines and points conception of figures, one can choose the variables so that construction is possible with a surfaces conception, but at a higher cost. The geometric use of instruments is gradually defined in conjunction with the definition of the cost of instruments. Many examples of figure restoration can be found in articles published by our research group, including Keskessa, Perrin-Glorian & Delplace (2007), Perrin-Glorian, Mathé & Leclercq (2013), Perrin-Glorian & Godin (2014), Mangiante & Perrin-Glorian (2018), Mathé, Barrier & Perrin-Glorian (2020), Mathé & Perrin-Glorian, (2023). Here is a simple example: we want to restore the figure consisting of a rectangle and its diagonals. One of the quarter rectangle triangles is given as a starting figure (Figure 5). We have three instruments to reproduce the figure: the ruler is free, each use of the length transfer is worth 2 points and the use of the compass is worth 1 point. How can we reproduce the figure with the least cost?

Figure 5 - Model figure and starting figure.



The starting figure is not on the same scale as the model and it is oriented differently; it is necessary to know which triangle it corresponds to: it can be checked by visual perception or by checking the angles with a folded sheet of paper, or the text indicates (for example, as here, by coloring it on the model) which triangle it is. With the choice made for the cost of the instruments, the winning strategy is to use the compass: we have the center of the circle circumscribed to the rectangle and two of its points, we can therefore draw it with the compass; it is then enough to extend the small sides of the isosceles triangle provided to obtain the diagonals and, at the intersection with the circle, the vertices of the rectangle and then draw the sides; the cost is 1 point. Another strategy is to transfer the length of the equal sides of the isosceles triangle to their support line from the top of the isosceles triangle (which is the intersection of the supports); it costs 4 points and requires using the property of diagonals to cut in their center. From any triangle you would get a parallelogram, but since you start from an isosceles triangle, the diagonals are equal, and you get a rectangle.

The first strategy is much more likely if the circumscribed circle appears on the model. Otherwise, it is necessary to have available the knowledge that a rectangle is inscribed in a circle whose center is the point of intersection of the diagonals. The transfer of the lengths with the compass can also bring out the circle: you do not need to change the spacing of the compass between the two transfers and you have the same center, so you draw two arcs of the same circle and only use the compass once to make the whole circle and get the missing vertices as the intersection of two lines (a straight line and a circle). Note that you could also restore the figure with a square. To block this procedure, the square is not allowed or is given an exorbitant cost.

In restoration, the geometric properties to be used for the construction must be recognized by the graphical properties of the model that can be verified with the instruments. In a construction, the conditions to be verified by the figure are given in a text and the properties must be available in the student's knowledge. Verification can be done by superimposing the solution on a tracing paper. Here, the fact that you get a rectangle and its diagonals can be verified using a square (four right angles). This can also be proved by the characteristic properties of the rectangle. The validation of the restoration can be done on several levels. We will come back to this.

The figure restoration allows the students to learn to enrich a figure with new lines that they can use in the restoration project: these new elements are related to those that they already have and to those that they want to obtain. In the example above, we can enrich the model by adding the circle. This activity thus helps to increase the semiotic thickness of the figures, which become carriers of information that is not immediately visible, and also that of the vocabulary: for example, the word "rectangle" will gradually carry relationships between its visible elements - angles, vertices, sides - and those of the enriched figure - diagonals, circumscribed circle, axes of symmetry, etc.

Figure restoration can target specific geometric properties (as in the example above), but it primarily targets learning related to geometric practice that is rarely considered explicitly, such as:

- A segment is supported by a straight line. A straight line can extend as far as you want on either side: once you know one segment of it, you know the whole line.
- You need two points to define a line.
- A point is created by the intersection of two lines.

These statements correspond to axioms or theorems derived more or less directly from axioms that are out of the question to prove with students. However, it seems important to us to identify with the teachers this knowledge, which is generally ignored in the teaching, but which is essential to allow the knowledge of theoretical geometry to be based on that of the geometry of tracing, overcoming the misunderstandings identified in numerous researches. For these learning purposes, we prefer to choose a figure that has no particular property (see an example in Mangiante and Perrin-Glorian, 2018).

Formulation and validation in the geometry of tracing

The conceptualization of geometric objects cannot be done only in action situations like the one we have described to illustrate the restoration of figures. It is also necessary to develop a language that makes it possible to describe the problems and the choices made to solve them, and also to provide ways to validate what is said in the language. (BULF, MATHE, MITHALAL, 2015). The theory of situations (BROUSSEAU, 1997) identifies three fundamental nested dialectics: action, formulation, validation. In action situations, there may be formulations, but they are not necessary. In formulation situations, they are made necessary by a constraint. Thus, in a situation of formulation about the restoration of a figure, the model and the starting figure are not available at the same time, for example by an exchange between sender and receiver: the sender has the model figure and the starting figure; he restores the figure, but his work does not stop there, he must write a message to a receiver who must restore the figure from the starting figure and the same instruments, but without having the model. Thus, it is necessary to formulate the methods used to reproduce the figure.

In the case of a figure restoration as an action situation, the validation is done using a tracing paper with the solution (different from the model figure in the case of a restoration at a different scale). This is physical evidence. In the validation situation, the goal is to move to intellectual evidence. This is a step towards proof that requires a new reading of the situation. (BALACHEFF, 2019, p. 14).

> This transition from pragmatic evidence to intellectual evidence, which is necessary in order to move towards proof, is also the transition from a pragmatic to a theoretical problematic, and thus an evolution of the reading of the situations in which mathematical activity unfolds and of the status of the knowledge mobilized (our translation).

Knowledge makes certain tests unnecessary. Gradually, a list of properties provided with the model figure or in response to questions asked by students after a first exploration of the model figure can replace the information taken from the model, and verification by tracing paper can be replaced by verification by instruments, even by reasoning. In the example above, why does the method used to restore the figure produce a rectangle with certainty, without the need to check it with tracing paper or with the instruments? First of all, it is necessary to make sure of the properties that can be taken for granted, the hypotheses, in this case that the triangle given as a starting figure is well isosceles. Then, whether we have transferred lengths or used the circle, it is the characterization of the rectangle by the diagonals that allows us to provide an intellectual proof that we obtain a rectangle. This requires, on the one hand, seeing the starting isosceles triangle and the resulting rectangle as generic examples representing all isosceles and rectangles connected by this construction, and, on the other hand, using known results to deduce others.

FROM THE GEOMETRY OF TRACING TO THE GEOMETRY OF STATEMENTS

Our approach aims at the coordination of three important dimensions: the mobility of the view on the material figures, the double role of the instruments (material and theoretical) and the geometric language, which uses the natural language and also symbols, for example coding, that can be introduced into the geometry of tracing to store information in a reproduction or a figure construction. The purpose of this paragraph is to try to clarify how figure restoration can help the entry into theoretical geometry, but also what separates them.

Figure restoration: first step to proof?

The process of restoring a material figure with tracing instruments aims to teach students to take a geometric view of figures through the need to analyze the visual characteristics of a figure in terms of properties that can be reproduced with instruments whose use, by the constraints we place on it, is close to the representation of geometric properties. It also aims to introduce a geometric language through its use in problem solving.

Table 1 – Comparison between restoration and proof in the use of the figure

Restoration	**Data:** Starting Figure, Model figure They are material figures (mute or with certain information).	**Instruments:** Material tracing instruments with their cost of use.	**Completed figure:** Conformity with the model is verified with a tracing paper (or with instruments for properties not used in construction). This verification requires either erasing some construction lines (or ignoring them) or redrawing the figure to be obtained.
		Knowledge: - Recognize in the model the starting figure and what makes it possible to connect it to the missing elements. Need to enrich the model figure. - Rules of use for the instruments to be linked to the desired tracings. - Organize the tracings in a certain order: the new tracings must be based on the tracings already obtained.	

Proof	**Data:** - Partial coding (by text or signs) of a theoretical geometric figure, sufficient to determine it. (Role played by the starting figure in the restoration). - A property of the figure to be proved. (Role played by the model figure).	**Instruments:** - Coded material figure (freehand or with instruments). - Geometrical knowledge (definitions, theorems). **Knowledge:** - Isolate the relevant elements on the material figure. - Identify key figures and link them to knowledge. - Organize the knowledge into an approach that links the data to the properties to be obtained.	**Completed coded figure:** Data and knowledge (theorems or definitions) have been used to construct and code the material figure that represents the theoretical figure. New properties have been proved: the code of the figure has been enriched.

Figure restoration also teaches students to establish a certain hierarchy between the properties that a figure verifies, some of which are already given (those provided by the starting figure), others that are identified on the model (which is enriched if necessary for this purpose), among which it will be necessary to select those that allow the reconstruction of the figure with the tools at hand, and others that are finally consequences of these. The search for this hierarchy of properties has a certain analogy with the path that connects hypotheses and conclusion in a proof. Table 1 shows a parallel between the two approaches in the use of the figure. The second column indicates the data in each case, the third column the work on the figure with the means that allow it, the last column, the result of this work and the means of validation. The big difference is that in the case of proof, the validation of properties is always done in language, using logical reasoning based on definitions and theorems, while in the case of restoration it can be done using material instruments.

As an example, let us compare a proof problem, a construction problem in the sense of theoretical geometry, and a restoration problem about the same figure.

Table 2 – Three problems about the same figure.

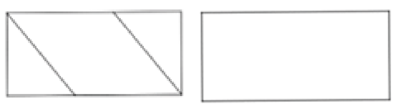
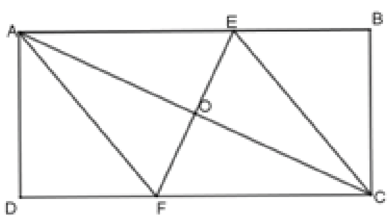
Problem 1. Let ABCD be a rectangle, E on [AB] such that EA=EC and F on [CD] such that FA = FC. Prove that AECF is a rhombus.	
Problem 2. Let ABCD be a rectangle. Construct a rhombus AECF such that E is on the segment [AB] and F on the segment [CD].	
Problem 3. Restore the figure. The rectangle is already traced. Instruments: ruler, square, compass.	

Figure 6 – Figure for solving the three problems.



In the first problem, the information is given in a text, we can realize the freehand figure to carry the information. The data call to mobilize the perpendicular bisector of [AC] or the isosceles triangles AEC and AFC. To show that AFCE is a rhombus, we must show, for example, that AE = AF, from which we deduce that the four sides have the same length. Depending on what we know, we can use the symmetry of the center O, the intersection of the diagonals of the rectangle, or show that the triangles EOC and AOF (or AEC and AFC) are equal.

In the second problem, the construction is not immediate: we are led to make the analysis of a freehand coded scheme carrying the properties to be obtained (problem supposed solved) to link them to those that it will be possible to use for the construction. In this analysis we are led to enrich the figure to look for other properties from which we can carry out the construction with the instruments we have. Here the analysis leads us to say that [AC] is a common diagonal for the rectangle and the rhombus. The diagonals of the rectangle and the rhombus intersect at their center, so [EF] passes through the center O of [AC]. Since the diagonals of a rhombus are perpendiculars, E and F necessarily lie at the intersections of the perpendicular to [AC] through O with the sides [AB] and [AC] of the rectangle. In theoretical geometry, however, it remains to prove that the condition is sufficient, i.e., that AECF is indeed a rhombus, which is done as in problem 1.

The third problem concerns the geometry of tracing. The points are not named, but for ease of writing we will name them as in Figure 6. Two segments are missing; to draw them, it is necessary to find two points (E and F) on the sides of the rectangle and, to do so, to enrich the model figure in order to connect the missing points to elements that can be drawn on the starting figure. The starting figure and the model are not on the same scale, so we cannot transfer lengths from the model to the starting figure; we have no measuring instrument, so we cannot use the numbers. The segment connecting the missing points is perpendicular to the diagonal of the rectangle, which is checked with the instruments. On the starting figure we can draw the diagonal of the rectangle. But in order to know where to draw the perpendicular, we must have checked on the model that [EF] passes through the center of [AC]. To find this center on the starting figure, we must draw the second diagonal of the rectangle using some knowledge: the diagonals of a rectangle intersect at their center. In the geometry of tracing, it is necessary to resort to a repertoire of knowledge (the proof of which is theoretical geometry, but which has the same statement). We check that we have a rhombus by comparing the lengths of the sides with the instruments.

Proof, geometric figure, and underlying axiomatics

To make proofs in theoretical geometry, we need definitions and theorems which, for the sake of consistency, must have an axiomatic basis, even if it is not formulated. We said in the introduction that the theory of Euclidean affine space is not enough to help teachers in their practice, they need a model of geometry based on sensitive space. This model was that of Euclid more or less adapted before the reform of modern mathematics. In France, in the 80s and 90s, the teaching of geometry was based on transformations, and cases of equality of triangles were ignored. For students, however, these are more accessible proof tools than transformations (PERRIN & PERRIN-GLORIAN, 2021). They require less dimensional deconstruction than transformations that most often require considering points that define lines and seeing a point as the intersection of two lines. For example, in Problem 1 in Table 2, to prove that AECF (Figure 6) is a parallelogram with central symmetry, the points E and F should be seen as the respective intersections of the line (EF) and the half-lines [AB) and [CD). The half-line [AB) has for image the half-line [CD), since A and B have for images C and D respectively; the line (EF) is transformed in itself, since it passes through O, so E has for image F and therefore AE = CF (so AE = AF). You must see a point as defined by the intersection of two lines and a line as defined by two points. To show that the triangles EOC and AOF are equal, we can use the case of isometry ACA (an angle between two sides): we know that AO = OC, the angles and are equal because they are right, the angles and are equal because alternate interior with respect to the secant (CA) of the parallels (AE) and (CF). Similarly, the isosceles triangles AEC and AFC have the AC side in common and the angles at the base and are equal because they alternate inside. The points and segments to be identified can be seen as vertices or sides of triangles, which is compatible with a view of figures as juxtapositions of surfaces. The choice of the underlying axiomatic therefore has important consequences for students' access to theoretical geometry.

USE IN THE CLASSROOM AND TEACHER TRAINING

The approach presented in this article was implemented in primary schools by a research group in the North of France between 2000 and 2010. The situations proposed are far from the usual practices of teachers. Collaborative research with primary school teachers (MANGIANTE-ORSOLA; PERRIN-GLORIAN, 2018; MANGIANTE-ORSOLA, 2023) has produced a resource (https://lea-geometrie.etab.ac-lille.fr/) aiming to help teachers to implement this approach and, in particular, trainers to develop training on the teaching of geometry in primary school and benchmarks of student progress. In collaboration with professors, research is underway to use figure restoration at the beginning of secondary education (between 11 and 13 years) to negotiate the transition to theoretical geometry.

CONCLUSION

Much research has focused on the rupture in the relationship to figures between a geometry based on perception, aided by tracing, or measuring instruments, and a deductive geometry, where it is a matter of validating, by proof from axioms theorems, and already established statements, about ideal theoretical objects. We have tried to bridge this gap by finding a missing link between these two geometries. To this end, we have defined what we have called a geometry of tracing, which concerns the objects of a graphic space that acts as an intermediary between the physical space of the material objects and the theoretical space of an axiomatic geometry, that relies solely on instruments for tracing and transferring magnitudes, excluding measurement by numbers. Our choice is justified because we focus our attention on the relationship to figures and the conceptualization of abstract geometric objects (essentially points and lines) on which rests an axiomatic of the type of that of Euclid. We have focused our reflection on the reproduction and construction of figures by identifying didactic variables that allow to design a progression (from 6 to 12 or 13 years), from the use of the outline of a template to reproduce a simple figure to the construction of figures with the ruler and the compass. One motivation for moving from simple drawing with instruments to the conceptualization of geometric objects seems to us to be the formulation of rules for the use of instruments around a central question: *which tracings already made do I need to place my instrument, which new tracing can be obtained?* This question prefigures an essential question of the proof process: *what are the data, what proposition do we want to prove?* In proof, theorems and definitions play the role that instruments played in the restoration or construction of a figure. To develop the geometric language, it is important that the situations proposed to the students are not limited to the dialectic of action, but also address the dialectic of formulation and the dialectic of validation, allowing to move from pragmatic proofs to first intellectual proofs using a repertoire of knowledge. Of course, there is still a long way to go to write proofs as expected at the end of secondary school.

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