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Binary Numeration: From Ancient Egypt to a 19th Century French Mathematical Recreation

A numeração binária: do Antigo Egito a uma recreação matemática francesa do século XIX

La numération binaire : de l'Égypte antique à une récréation mathématique du XIXe siècle en France

La numeración binaria: desde el Antiguo Egipto hasta una recreación matemática Francés del siglo XIX

Marc Moyon[1]

ABSTRACT

This article focuses on the use of base 2 in two historical contexts: ancient Egypt during the Middle Kingdom and 19th-century France. In ancient Egypt, base 2 was central to the "duplation" algorithm to multiply two numbers. In 19th-century France, it featured in mathematical recreations by Charles-Ange Laisant and Édouard Lucas. Duplation in ancient Egypt involved doubling a number successively, leveraging the binary nature of the numbering system for rapid calculations. In France, Laisant and Lucas contributed to mathematical recreations rooted in base 2, popularizing mathematics and engaging the public. The article aims to present these historical resources, highlighting their educational potential. It suggests integrating these examples into mandatory education from age 9 onwards, offering students a practical understanding of mathematics and its history. However, specific teaching methods are left to the reader's discretion.
Keywords: History of Mathematics; Mathematical Recreations; Mathematics Education; Binary system; Numbers.

RESUMO

Este artigo tem como foco o uso da base 2 em dois contextos históricos: o antigo Egito durante o Império Médio e na França do século XIX. No antigo Egito, a base 2 era essencial para o algoritmo de multiplicação, conhecido como "duplicação". Entretanto, no século XIX na França, apareceu em recreações matemáticas apresentadas por Charles-Ange Laisant e Édouard Lucas. No antigo Egito a duplicação implicava duplicar sucessivamente um número, aproveitando a natureza binária do sistema de numeração para realizar cálculos rápidos. Na França, Laisant e Lucas contribuíram às recreações matemáticas fundadas na base 2, o que ajudou a popularizar as matemáticas e a envolver o público. O artigo tem como objetivo apresentar estes recursos históricos, ressaltando seu potencial educativo. Sugere integrar estes exemplos na educação obrigatória a partir dos 9 anos, proporcionando aos estudantes uma compreensão prática das matemáticas e sua história. Sem dúvida, os métodos de ensino específicos ficam a critério do leitor.
Palavras-chave: História da Matemática; Sistema Binário; Números; Recriações Matemáticas; Educação Matemática.

RÉSUMÉ

Cet article met en lumière l'utilisation de la base 2 dans deux contextes historiques distincts : l'Égypte antique pendant le Moyen-Empire et la France du XIXe siècle. Dans l'Égypte ancienne, la base 2 était au cœur de l'algorithme de multiplication connu sous le nom de « duplication ». En France, au XIXe siècle, elle était entre autres utilisée dans une récréation mathématique présentée par deux mathématiciens : Charles-Ange Laisant et Édouard Lucas. La méthode de la duplication, utilisée dans l'Égypte antique, consistait à multiplier un nombre en le doublant successivement, exploitant ainsi la nature binaire du système de numération. Cette technique permettait des calculs efficaces et rapides. En France, au XIXe siècle, Charles-Ange Laisant et Édouard Lucas ont participé au développement de récréations mathématiques basées sur la base 2. Ces jeux et énigmes ont contribué à populariser les mathématiques et à susciter l'intérêt du grand public pour ce domaine. Une de celles-ci, l'éventail mystérieux, est présentée dans cet article. L'article vise à exhumer ces ressources historiques dans leur contexte respectif, en mettant l'accent sur leur utilisation pédagogique potentielle. Il suggère que ces exemples pourraient être intégrés dans l'enseignement obligatoire dès l'âge de 9 ans, offrant ainsi aux élèves une perspective concrète sur l'histoire des mathématiques. Cependant, l'approche pédagogique précise est laissée à l'appréciation du lecteur, qui est encouragé à explorer différentes méthodes pour intégrer ces concepts dans son enseignement.
Mots-clés: Histoire des mathématiques; Récréations mathématiques; Éducation mathématiques; Système binaire; Nombres.

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Fluxo Contínuo

RESUMEN

Este artículo se enfoca en el uso de la base 2 en dos contextos históricos: el antiguo Egipto durante el Reino Medio y en Francia del siglo XIX. En el antiguo Egipto, la base 2 era esencial para el algoritmo de multiplicación conocido como "duplicación". Mientras tanto, en el siglo XIX en Francia, se incorporó en recreaciones matemáticas por Charles-Ange Laisant y Édouard Lucas. La duplicación en el antiguo Egipto implicaba duplicar sucesivamente un número, aprovechando la naturaleza binaria del sistema de numeración para realizar cálculos rápidos. En Francia, Laisant y Lucas contribuyeron a recreaciones matemáticas basadas en la base 2, lo que ayudó a popularizar las matemáticas y a involucrar al público. El artículo tiene como objetivo presentar estos recursos históricos, resaltando su potencial educativo. Sugiere integrar estos ejemplos en la educación obligatoria a partir de los 9 años, proporcionando a los estudiantes una comprensión práctica de las matemáticas y su historia. Sin embargo, los métodos de enseñanza específicos quedan a discreción del lector.
Palabras clave: Historia de las Matemáticas; Recreaciones Matemáticas; Educación Matemática; Sistema Binario; Números.

INTRODUCTION

In this article detailing a presentation delivered in Bordeaux (France) in June 2023[2], I delve into two pivotal moments in the history of mathematics, tracing back to the utilization of binary numeration – the Middle Kingdom of Egypt and late 19th-century France. I believe these epochs offer valuable pedagogical insights for classroom exploitations. Moreover, the geographical and temporal breadth covered herein presents a compelling narrative, juxtaposing two disparate realms across vast epochs seemingly disconnected from one another.

To begin, I present select excerpts from the renowned Rhind mathematical papyrus (1800 BCE), followed by an exploration of the scholarly endeavors of two French mathematicians from the late 19th century: Charles-Ange Laisant (1841-1920) and Édouard Lucas (1842-1891). Their collaborative efforts significantly influenced French discrete mathematics at the close of the century and left an indelible mark on the realm of recreational mathematics. This provides a unique opportunity to unveil hitherto unpublished sources, including snippets from the correspondence exchanged between these distinguished scholars.

Central to my inquiry is the concept of binary numeration, which serves as a common thread bridging Egyptian mathematical practices with those of the French. This numerical system, championed notably by Gottfried Wilhelm Leibniz (d. 1716) and his seminal work[3], *De Progressione Dyadica* (manuscript dated March 15, 1679), found fertile ground for the development of mathematical curiosities, recreational pursuits, and even magical feats, as we shall soon explore. The pedagogical value of recreational mathematics is unequivocal, fostering curiosity and engagement among learners while often requiring hands-on manipulation to deepen conceptual understanding.

THE DUPLATION IN THE RHIND MATHEMATICAL PAPYRUS

In the Middle Kingdom of Egypt (during the era of pyramid construction), scribes conducted multiplication solely through the method of doubling, known as duplation. This technique was also employed for division (RITTER, 2000, p. 126), may be called mediation. Additionally, it persisted in practice and instruction until, at least, the 16th century AD due to its suitability for abacus calculations (CAVEING, 1994, p. 253; STRICKLAND, 2022). It involves breaking down a number into a sum of non-negative integer powers of 2. While such a decomposition is feasible for all integers, it may not always be unique. In his *Essai sur le*

2 See Moyon (2024)
3 Serra (2017) provides an in-depth examination of Leibniz's manuscript.

savoir mathématique dans la Mésopotamie et l'Égypte anciennes[4], Caveing elucidates the "*fundamental theorem* of Egyptian arithmetic":

> Let's consider the infinite sequence of increasing powers of 2: $2^0, 2^1, 2^2, \ldots, 2^k, \ldots$ For any natural number, it either belongs to the list of terms in this sequence or is the sum of terms that appear in it[5] (CAVEING, 1994, p. 253-258).

If the Egyptian scribes manipulated this result, none had explicitly stated it as a property or provided a demonstration. The Rhind mathematical papyrus derives its name from its purchaser, the Scottish lawyer Alexander Henry Rhind, in 1858. It is currently housed in the British Museum and stands as one of the most pivotal documents for comprehending Egyptian mathematics.

In problem #32, the scribe Ahmes is tasked with determining the quantity that, when augmented by both its third and its quarter, yields 2 (in modern writing, with Algebra, we have: $x + \frac{1}{3}x + \frac{1}{4}x = 2$). In the course of solving this problem, the scribe is led to compute the product of 12 by 12 (Fig. 1). In this instance, deciphering the hieroglyphs of the operation (which employs additive decimal numeration, with the staff representing units, the handle representing tens, and the coiled rope representing hundreds) is not overly challenging. Drawing from personal experience, it's feasible for a student as young as 9-10 years old to grasp the significance of lines 18 to 22, explaining that the symbol transcribed as *dmḏ* may mean *sum*[6].

Let's revisit the first four lines. (Tab. 1 below).

Table 1 – Explanation of the initial lines of the multiplication of 12 by 12

.	12	Let's consider 12, one of the two factors of the product (often the larger of the two).
2	24	Let's double 12.
4	48	Let's double 24.
8	96	Let's double 48.

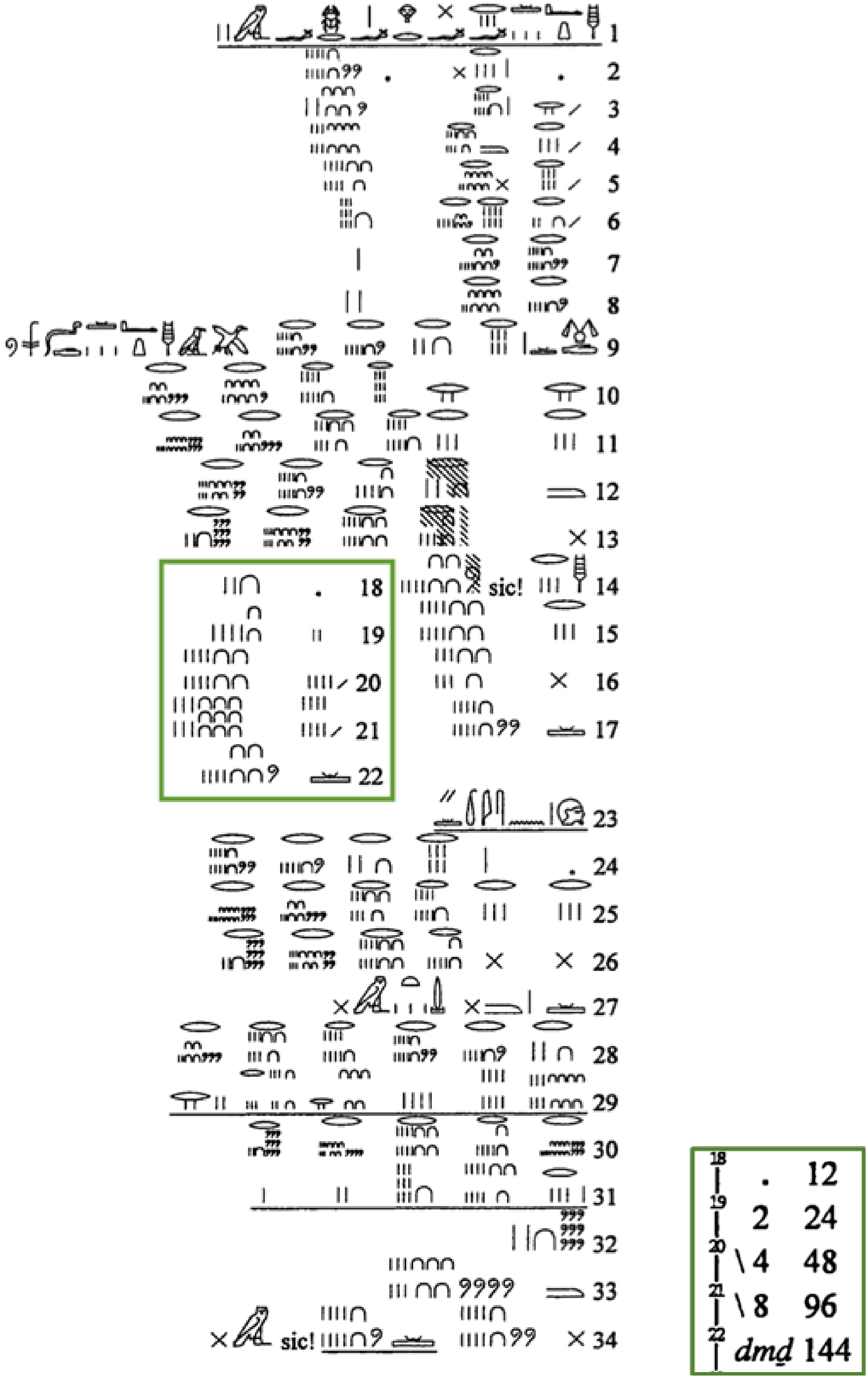
Source: Personal elaboration

4 All translations are mine (even in appendices). For primary source material I have retained the original French in footnotes (unless in appendices).

5 *Soit la série non-limitée des puissances croissantes de 2 : $2^0, 2^1, 2^2, \ldots, 2^k, \ldots$, tout entier naturel ou bien figure dans la liste des termes de cette série, ou bien est la somme de termes qui y figurent.*

6 Imhausen translates *dmḏ* by the German word *summe*. Michel (2014, p. 72) points out that the term can also be used as a verb with the meaning of *add*.

Figure 1 – Problem #32 of the Rhind papyrus, revealing a multiplication (lines 18 to 22)



Source: Text (on the left) and transcription (on the right) from Imhausen (2003, p. 216-217)

It is now necessary to explain the fifth line, named 'sum' (Tab. 2).

Table 2 – Explanation of the multiplication of 12 by 12

.	12	Let's consider 12
2	24	Let's double 12, we obtain 24.
\4	48	Let's double 24, we obtain 48.
\8	96	Let's double 48, we obtain 96.
sum	144	We compute 144 as the sum of 48 and 96, by adding the digits 4 and 8 from the first number (marked with

Source: Personal elaboration

Mathematically, the scribe Ahmes uses the distributivity of multiplication over addition. Expressed here in modern terms, we have:

$$12 \times 12 = (4 + 8) \times 12 = 4 \times 12 + 8 \times 12 = 48 + 96$$

Furthermore, by systematically doubling at each line, the scribe manages to formulate a decomposition of the multiplicand into a sum of powers of 2 (Tab. 3).

Thus, $12 = 2^2 + 2^3 = 0 \times 2^0 + 0 \times 2^1 + 1 \times 2^2 + 1 \times 2^3$.

And, $1\ 1\ 0\ 0$ is then the representation of 12 in base 2 (or in binary notation).

Table 3 – First powers of 2.

1	2^0
2	2^1
4	2^2
8	2^3
16	2^4
…	$2^{\dots}$

Source: Personal elaboration

The method employed by Egyptian scribes for multiplying two numbers can be succinctly described as follows: one of the two numbers, typically the smaller one (the multiplicand), is decomposed into a sum of powers of two. The other number (the multiplier) is then systematically doubled according to the corresponding power of 2. Subsequently, the distributive property is implicitly utilized to compute a sum. While alternative approaches exist where the decomposition involves numbers other than 2 (or its powers) – such as 10, for instance – they are typically tailored to specific problem contexts rather than being universally applicable[7]. Moreover, Egyptian scribes relied solely on memorizing the multiplication table for 2; all other multiplication tables were deemed redundant, albeit with a few exceptions. This historical observation intersects with its pedagogical implications[8], thanks to "its dual character of generality and comparative simplicity, juxtaposed with the laborious

7 For illustrative examples, refer to Imhausen (2016, p. 86-88).

8 Also noteworthy is its scientific relevance. Presently, computer scientists continue to derive benefits from this method, notably in the context of the 'Exponentiation by squaring' (an algorithm used to compute large integer powers, employed, among other fields, in cryptography).

task of committing to memory the multiplication table requisite for the decimal system[9]" (CAVEING, 1994, p. 253).

Table 4 – The multiplication of 3 by 5 in the problem #25 of the Rhind papyrus.

$\backslash$.	3	Let's consider 3, the initial number.
2	6	Let's double 3, we obtain 6
$\backslash$ 4	12	Let's double 6, we obtain 12
sum	15	$5 = 1 + 4$, hence $3 \times 5 = 3 + 12 = 15$

Source: Personal elaboration

For products involving small factors (Tab. 4), the algorithm may appear longer than using multiplication tables (like us today). However, as soon as the factors become larger (as seen, for example, in Tab. 5), the method quickly becomes efficient.

Table 5 – The multiplication of 59 by 63.

$\backslash$.	63	Let's consider 63.
$\backslash$ 2	126	Let's double 63, we obtain 126.
$\backslash$ 4	252	Let's double 126, we obtain 252.
$\backslash$ 8	504	Let's double 252, we obtain 504.
$\backslash$ 16	1 008	Let's double 504, we obtain 1 008.
$\backslash$ 32	2 016	Let's double 1 008, we obtain 2 016.
sum	3 717	$59 = 1+2+8+16+32$, hence $59 \times 63 = 63 + 126+504+1\ 008+2\ 016=3\ 717$

Source: Personal elaboration

Furthermore, this method finds application in computing products involving an integer and a fraction, or between fractions, as exemplified in the problems #27 and #70 of the Rhind papyrus (Tab. 6). The scribe employs elementary fractions such as $\frac{1}{2}$ or $\frac{2}{3}$ (as evidenced in the fifth row of Tab. 6, on the right), whether from rote memorization or through intermediary computations. In ancient Egypt, fractions were consistently represented as sums of unit fractions (except for the fraction $\frac{2}{3}$, which had its distinct representation). In (MOYON, 2023b), the discourse delves into the realm of Egyptian fractions within mathematical pedagogy and their treatment within French mathematical textbooks.

9 *son double caractère de généralité et de relative simplicité comparativement à l'acquisition laborieuse par la mémoire de la table de multiplication nécessaire au système décimal.*

Table 6 – (left) $5 \times \left(3+\frac{1}{2}\right)$ in the problem #27; (right) $\left(12+\frac{2}{3}\right) \times \left(7+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}\right)$ in the problem #70.

\\.	$3+\frac{1}{2}$
2	7
\\ 4	14
sum	$17+\frac{1}{2}$

.	$7+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}$
2	$15+\frac{1}{2}+\frac{1}{4}$
\\ 4	$31+\frac{1}{2}$
\\ 8	63
\\ $\frac{2}{3}$	$5+\frac{1}{4}$
sum	$99+\frac{1}{2}+\frac{1}{4}$

Source: Personal elaboration

A MATHEMATICAL OUTLOOK EMBRACED BY BOTH CHARLES-ANGE LAISANT AND ÉDOUARD LUCAS

I do not claim here to provide new insights into the lives and works of Laisant and Lucas, two French mathematicians from the late 19th century. However, it seems important to offer some elements to reveal the connections between the two mathematicians. This also provides an opportunity to shed light on an active French community of mathematicians[10] and to exhibit unpublished letters and other documents from several archives.

Édouard Lucas (1842-1891)

The biographical information about Édouard Lucas is drawn from the research conducted by Anne-Marie Décaillot (1998, 1999). As a prominent French mathematician of the late 19th century, Lucas is regarded as one of the most prolific scholars in the field of number theory during this era (DÉCAILLOT, 1998, 191). Initially a student at the *École Normale Supérieure* (Paris), Lucas later embarked on a career as an astronomer at the *Observatoire* of Paris. Subsequent to the Franco-Prussian War of 1870, he assumed several professorships in mathematics, first in provincial towns such as Moulins, and later in Paris at the *Lycée Charlemagne* and the *Lycée Saint Louis*.

Lucas's mathematical endeavors primarily centered on number theory, and he became particularly renowned for his contributions to a primality test. This test, later refined by the English mathematician Lehmer in the 1930s, is now known as the Lucas-Lehmer test. Tragically, Lucas met an untimely demise in October 1891, succumbing to injuries sustained during an accident at a banquet held during the Congress of the *Association pour l'Avancement des Sciences* (AFAS) in Marseille. His passing left behind a legacy of unfinished works.

Lucas authored several articles and books during his lifetime, including his *Théorie des nombres* (1891a), for which only the first volume was published. Some of his works, such as the last two volumes of their *Récréations mathématiques* (1891b) and his *Arithmétique*

10 About French mathematics in the 19th century around Lucas, Laisant and others, read (GOLDSTEIN, 2020).

amusante (1895), were published posthumously thanks to the work of three of his French colleagues. Additionally, he authored numerous mathematical games in collaboration with the manufacturer Chambon & Baye (Fig. 2), earning him recognition at the *Exposition universelle* in Paris, in 1889 (appendix 1).

Figure 2 – Presentation card for Lucas' games

Source: BnF NAF (Nouvelle acquisition française) n°28336 (fonds Laisant), 115

This is evidenced by the following review (appendix 1) published in the daily newspaper *La France*, where the reviewer emphasizes:

> the frivolity of these games is only apparent. They sometimes touch upon the highest considerations of the science of combinations, and are, of a nature, in all cases, to make the study of calculation attractive, to develop taste, vision, and skill, especially concerning geometric drawing.
>
> Educational establishments and families would be greatly mistaken to deprive themselves of such a powerful pedagogical tool. [...] (GAEL, 1889, p. 3).

Charles-Ange Laisant (1841-1920)

Charles-Ange Laisant remains a relatively obscure mathematician of the late 19th century, despite several biobibliographical works dedicated to him, notably the extensive research by Jérôme Auvinet, such as his monograph (AUVINET, 2013) entirely devoted to him. In particular, Auvinet adeptly illustrates the various scientific and political role played by this mathematician[11], a graduate of the *École Polytechnique* (X1859), born near Nantes in 1841.

11 Initially serving as a councilor in Nantes, C.-A. Laisant later became a deputy representing the department of *Loire-Inférieure* (1876-1885) and then the *Seine* (1885-1893).

Charles-Ange Laisant is associated with French mathematicians such as Édouard Lucas, Émile Lemoine (1840-1912, X1860), Henri Auguste Delannoy[12] (1833-191, X1853), and also with the Belgian number theorist Eugène Charles Catalan[13] (1814-1894). He delves into discrete mathematics (AUVINET, 2017), particularly in their applications to mathematical recreations. Specifically, he collaborates with Delannoy and Lemoine in posthumously editing the last two volumes of Lucas' *Récréations mathématiques* (DÉCAILLOT, 2014), as well as his *Arithmétique amusante*. Laisant also aspired to publish a portion of Lucas's correspondence. Hence, he writes to Delannoy in a letter dated October 11, 1891:

> I've just taken a look at my correspondence with Lucas, of which I've kept the letters. Don't you think it would be a good idea for all friends to do the same and, if necessary, to issue a call on this subject to a few of them? It seems to me that there would be many interesting things to extract from this correspondence, where his lively spirit gave free rein and generously offered ideas from which mathematicians of the future could benefit[14]. (Letter from Laisant to Delannoy, *Archives départementales de la Creuse*[15], fonds Henri Delannoy, 10F)

As an editor, an active member of prestigious societies – such as the *Société Mathématique de France* (SMF), where he was elected president in 1888, and the AFAS (similar to Lucas), among others – or even as a politician, Charles-Ange Laisant is a well-connected individual who tirelessly advocates for the advancement of sciences, with a particular focus on mathematics, catering to all demographics. He is deeply committed to producing mathematical journals with broad appeal. Consequently, Laisant founded and oversaw the operation of various mathematical publications both domestically and internationally, including the *Nouvelles Annales de Mathématiques*, *L'Intermédiaire des mathématiciens* (with Émile Lemoine), and *L'enseignement mathématique* (with Henri Fehr), each tailored to diverse audiences (EHRHARDT, 2018).

Charles-Ange Laisant is well known in Brazil, as attested by the recent study by Circe Mary Silva da Silva and Maria Célia Leme da Silva:

> The educational works of Laisant, geared towards general teaching and particularly mathematics, found swift recognition in Brazil, both in their original French editions and in translated versions in Portuguese. [...] Laisant's pedagogical concepts swiftly permeated Brazilian intellectual circles through mainstream publications, mathematics textbooks, and educational journals from 1906 to 1957. [...] Brazilian educators particularly embraced Laisant's ideas on several fronts: emphasizing the importance of drawing in education, recognizing the pivotal role of hands-on experience in knowledge acquisition, advocating for the integrated teaching of arithmetic, geometry, and algebra, promoting an intuitive and experimental introduction to geometry, and, most importantly, fostering a more engaging and playful approach to mathematics education accessible to all[16] (SILVA DA SILVA; LEME DA SILVA, 2023, p. 152).

12 About Delannoy and more generally about the mathematics of Laisant , read the collective work (BARBIN & *al.*, 2017).

13 About Catalan, read (BOUCARD, 2015).

14 *Je viens de jeter un coup d'œil à ma correspondance avec Lucas, dont j'ai conservé les lettres. Ne pensez-vous pas qu'il serait bon que tous les amis en fissent autant et qu'on adressât au besoin un appel à ce sujet à quelques uns d'entre eux ? Il me semble qu'il y aurait beaucoup de choses intéressantes à extraire de cette correspondance, où son esprit si primesautier se donnait libre cours et lançait à pleines mains des idées dont les mathématiciens de l'avenir pourraient tirer profit* […].

15 https://archives.creuse.fr/ark:22575/1ee8e8f82e7761b6865c0050568b525d.fiche=arko_fiche_612f95c300275.moteur=arko_default_60d08acd47b45

16 *Les œuvres de Laisant destinées à l'enseignement en général et à celui des mathématiques en particulier, à la fois en français et traduites en langue portugaise, ont été rapidement connues au Brésil. […] La diffusion des idées pédagogiques de Laisant s'est opérée dans la presse généraliste, les manuels de mathématiques et les revues pédagogiques au Brésil de 1906 à 1957. […] Les principales appropriations de Laisant par les auteurs brésiliens ont été : la valorisation de l'enseignement du dessin ; le rôle important de l'expérience dans la construction des connaissances ; l'enseignement intégré de l'arithmétique, de la géométrie et de l'algèbre ; une initiation intuitive et expérimentale à la géométrie et, surtout, un enseignement des mathématiques plus ludique pour tous.*

This last quotation precisely emphasizes the vision of mathematics defended by Laisant and shared with Lucas.

Their view of mathematics

In other works (MOYON, 2023a, 2023c), I have discussed the ideas advocated by Laisant for mathematics education. In particular, Laisant vehemently opposes the notion, which can be summarized as follows: "to expect everything from memory and almost nothing from intelligence"[17] (LAISANT, 1907, p. 6). This opposition forms one of his many personal battles. While Laisant authored numerous articles and mathematical works, he also penned numerous contributions on mathematics education, particularly elementary education, emphasizing manipulation and visualization to lead all children to abstraction and understanding of mathematical concepts, even the most abstract ones. Furthermore, he continues in *L'enseignement du Calcul* (1920), a small pamphlet intended for teachers:

> And this, because the psychology of the young child is very special, notably very rebellious to dogmatic formulas. [...] Thus, [the teachers] will form, not little parrots, who arrive at reciting their lessons through a true torture of memory, but well-balanced children: they will not be stuffed with artificial knowledge, they will understand what they have learned; they will never learn without understanding[18] (LAISANT, 1920, p. 1-2).

He seems to be joined in his main pedagogical ideas by his correspondent Édouard Lucas as we can read in a undated letter addressed to him (appendix 2). Among other things, one recognizes the importance of manipulation (use of fingers or tokens), awakening curiosity with the use of amusements. Lucas also supports the analytical method without resorting too quickly to definitions, thus avoiding any dogmatism.

In my final part, I will detail one of the numerous mathematical recreations proposed by Lucas and Laisant in their respective works.

A mathematical recreation: The *éventail mystérieux* [mysterious fan]

As an extension of the discovery of the duplation method among the Egyptians, it can be both instructive and delightful to explore activities centered around binary numeration – that is, in base 2 – represented by the symbols 0 and 1.

As previously observed, the binary representation of 12 is 1 1 0 0, given that:

$$12 = 2^2+2^3 = 0 \times 2^0 + 0 \times 2^1 + 1 \times 2^2 + 1 \times 2^3.$$

Similarly, the binary expression of 59 is 1 1 1 0 1 1.

$$59 = 1 \times 2^0 + 1 \times 2^1 + 0 \times 2^2 + 1 \times 2^3 + 1 \times 2^4 + 1 \times 2^5$$

In his *Récréations mathématiques*, Lucas delineates at least two advantages of binary numeration (LUCAS, 1891b, p. 148). Firstly, he notes that "ordinary arithmetic operations are simplified to their most basic form[19]". Secondly, he highlights that binary numeration

17 *tout attendre de la mémoire et presque rien de l'intelligence.*

18 *Et cela, parce que la psychologie du petit enfant est très spéciale, notamment très rebelle aux formules dogmatiques. [...] C'est ainsi [que les instituteurs] formeront, non pas de petits perroquets, arrivant à réciter leurs leçons grâce à une véritable torture de la mémoire, mais des enfants bien équilibrés : ils ne seront pas bourrés de connaissances factices, ils comprendront ce qu'ils ont appris ; ils n'apprendront jamais sans comprendre.*

19 *les opérations ordinaires de l'arithmétique sont réduites à leur expression la plus simple.*

enabled him to discover prime numbers significantly larger than those previously known. However, he also concedes the significant drawback of having to employ a considerable number of symbols (0 and 1) to represent a large number.

Considering the importance of binary numeration in understanding numbers and numeral systems, I have chosen to present a “small game” inspired by the Polytechnician Charles-Ange Laisant: “a simple salon game built around binary numeration, described by Éd. Lucas in his *Arithmétique amusante* as the *éventail mystérieux*”[20] (LAISANT, 1915, p. 106), which can be easily adapted for classroom use.

The *Arithmétique amusante*, signed by Lucas, was actually edited posthumously by Laisant, Delannoy, and Lemoine (as I wrote before) in 1895, from three notebooks prepared during his lifetime, starting in 1888. Let's then read Édouard Lucas detail the rules of the game

> The *éventail mystérieux* consists of cards arranged in a fan shape, on which numbers are written […] in a certain way; the aim, when presenting the fan, is to guess the number […] thought of by a person[21] (LUCAS, 1895, p. 168-169).

Figure 3 – Cards from the *éventail mystérieux*

A	B	C	D	E
1	2	4	8	16
3	3	5	9	17
5	6	6	10	18
7	7	7	11	19
9	10	12	12	20
11	11	13	13	21
13	14	14	14	22
15	15	15	15	23
17	18	20	24	24
19	19	21	25	25
21	22	22	26	26
23	23	23	27	27
25	26	28	28	28
27	27	29	29	29
29	30	30	30	30
31	31	31	31	31

Source: Laisant (1915, p. 107)

If we follow Lucas in his *Arithmétique amusante*, we must (1) guess a number thought of by a player and (2) understand how the numbers are written on the cards.

1. Let's take an example: the number 27. If we know in which column(s) it appears (and it's enough to ask the player who thought of the number), it's easy to find it. Indeed, here, 27 appears in columns A, B, D, and E: all we have to do is “to sum the numbers written at the top of each of

20 *un petit jeu de salon qui repose sur l’emploi de la numération binaire, et dont Ed. Lucas a reproduit la description dans son Arithmétique amusante sous le nom d’Eventail mystérieux.*

21 *L’éventail mystérieux se compose de cartons disposés en éventail sur lesquels on inscrit des nombres […] d’une certaine manière ; il s’agit, en présentant l’éventail, de deviner le nombre […] pensé par une personne.*

the cards where the number is found"[22] (LUCAS, 1895, p. 169). So, let's add 1, 2, 8, and 16; the sum is 27; it's indeed the number thought of!

2. The five cards in Fig. 3 contain all the numbers from 1 to 31 according to their binary representation (or their decomposition into sums of powers of 2). Let's take again the example of the number 27, as $27 = 1 \times 2^0 + 1 \times 2^1 + 0 \times 2^2 + 1 \times 2^3 + 1 \times 2^4$, it is written as: 1 1 0 1 1 in binary notation and it then appears in the first (A), second (B), fourth (D), and fifth (E) columns (where the binary representation gives 1) and does not appear in the third column (C) (where the binary representation gives 0).

Let's now take the number 23:

$$23 = 1 \times 2^0 + 1 \times 2^1 + 1 \times 2^2 + 0 \times 2^3 + 1 \times 2^4$$

Hence the binary writing of 23 is 1 0 1 1 1 : it is in the columns A, B, C and E.

Playing the game untitled *éventail mystérieux* presents a wonderful opportunity to engage students with arithmetic in the classroom, much like with other arithmetic games: various magic tricks and puzzles, such as the *tours de Hanoï* [Tower of Hanoi] or the *Baguenaudier* [Chinese rings] (ROUGETET, 2023).

However, Laisant also suggests "expanding this game up to 63 instead of 31, using 6 cards instead of 5, and up to 127 with 7 cards"[23] (LAISANT, 1915, p. 110). This idea, while ambitious, is compelling as it prompts students to delve deeper into binary code, an essential component of all digital technologies. The extension of the *éventail mystérieux* proposed by Laisant (Fig. 5) then includes six cards (from 1 to 63) instead of five (from 1 to 31).

22 *de faire la somme des nombres écrits en tête de chacun des cartons où le nombre se*
23 *pousser ce jeu jusqu'à 63 au lieu de 31, avec 6 cartons au lieu de 5, et jusqu'à 127 avec 7 cartons.*

Figure 4 – Extension of the *éventail mystérieux* to 63 (instead of 31)

1	2	4	8	16	32
3	3	5	9	17	33
5	6	6	10	18	34
7	7	7	11	19	35
9	10	12	12	20	36
11	11	13	13	21	37
13	14	14	14	22	38
15	15	15	15	23	39
17	18	20	24	24	40
19	19	21	25	25	41
21	22	22	26	26	42
23	23	23	27	27	43
25	26	28	26	28	44
27	27	29	29	29	45
29	30	30	30	30	46
31	31	31	31	31	47
33	34	36	40	48	48
35	35	37	41	49	49
37	38	38	42	50	50
39	39	39	43	51	51
41	42	44	44	52	52
43	43	45	45	53	53
45	46	46	46	54	54
47	47	47	47	55	55
49	50	52	56	56	56
51	51	53	57	57	57
53	54	54	58	58	58
55	55	55	59	59	59
57	58	60	60	60	60
59	59	61	61	61	61
61	62	62	62	62	62
63	63	63	63	63	63

Source: Personal elaboration

CONCLUSION

Rather than embarking on a new empirical study, as delineated by Jankvist (2009b) and echoed by Guillemette (2011), my aim was to spotlight two distinct historical junctures in mathematics, seemingly disparate, yet both bearing relevance to the study of binary numeration. Through the examination of Egyptian duplation methods and the mathematical recreations of Laisant-Lucas, I argue for the convergence of history and pedagogy, and potentially didactics, within this domain. These historical vignettes serve as valuable resources for an array of learning scenarios, to be crafted by educators or facilitators, fostering students' appreciation for mathematics' societal contributions (FAUVEL and VAN MAANEN, 2000, p. 1-29).

Drawing from Jankvist's framework (2009a), I posit that these exemplars align with a modular learning approach[24], rather than simply serving as anecdotal or historical vignettes, the didactic merit of which may be subject to debate. History, in this context, illuminates pivotal mathematical concepts, potentially leveraging primary sources such as manuscripts, correspondence, or archival materials.

The examples elucidated in this study serve a multiplicity of functions: (1) illuminating pre-existing mathematical constructs and writings, thus showcasing the progressive evolution of the discipline; (2) offering access to a diverse array of topics and challenges; (3) deepening comprehension of mathematical concepts; (4) humanizing mathematics; and (5) underscoring the ubiquitous nature of mathematics across diverse contexts and environments.

As expounded by Rougetet (2023, p. 111-120), mathematical recreations – whether contemporary or historical – offer valuable insights for mathematics education, owing to their inherently playful nature. The *éventail mystérieux* devised by Laisant-Lucas stands as a compelling exemplar within this broader context.

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24 Jankvist refered to the "modules approaches" following Kats and Michalowicz (2004).

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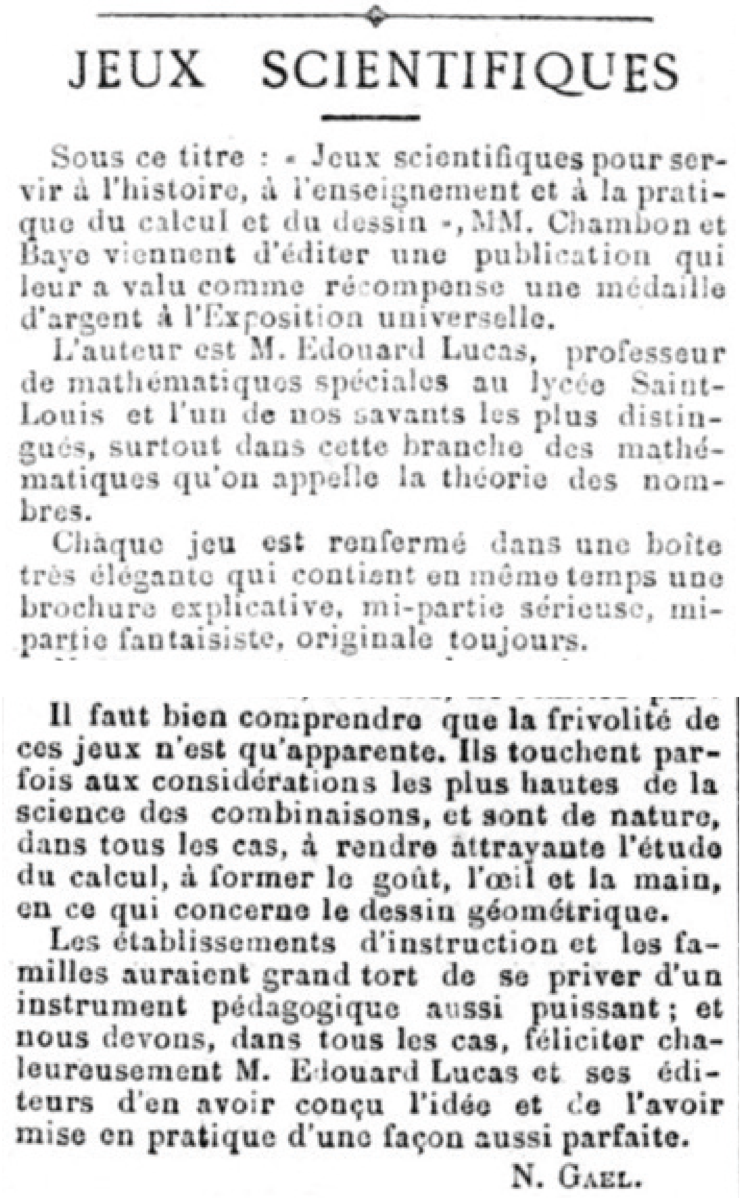
Appendix 1 - Excerpt from the section untitled *Jeux Scientifiques* [scientific games] of the French daily newspaper *La France*



JEUX SCIENTIFIQUES

Sous ce titre : « Jeux scientifiques pour servir à l'histoire, à l'enseignement et à la pratique du calcul et du dessin », MM. Chambon et Baye viennent d'éditer une publication qui leur a valu comme récompense une médaille d'argent à l'Exposition universelle.

L'auteur est M. Edouard Lucas, professeur de mathématiques spéciales au lycée Saint-Louis et l'un de nos savants les plus distingués, surtout dans cette branche des mathématiques qu'on appelle la théorie des nombres.

Chaque jeu est renfermé dans une boîte très élégante qui contient en même temps une brochure explicative, mi-partie sérieuse, mi-partie fantaisiste, originale toujours.

Il faut bien comprendre que la frivolité de ces jeux n'est qu'apparente. Ils touchent parfois aux considérations les plus hautes de la science des combinaisons, et sont de nature, dans tous les cas, à rendre attrayante l'étude du calcul, à former le goût, l'œil et la main, en ce qui concerne le dessin géométrique.

Les établissements d'instruction et les familles auraient grand tort de se priver d'un instrument pédagogique aussi puissant ; et nous devons, dans tous les cas, féliciter chaleureusement M. Edouard Lucas et ses éditeurs d'en avoir conçu l'idée et de l'avoir mise en pratique d'une façon aussi parfaite.

N. Gael.

Source: *La France*, 02/12/1889, p.3

French text

Sous ce titre : « jeux scientifiques pour servir à l'histoire, à l'enseignement et à la pratique du calcul et du dessin », MM. Chambon et Baye viennent d'éditer une publication qui leur a valu comme récompense une médaille d'argent à l'exposition universelle.

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Les établissements d'instruction et les familles auraient grand tort de se priver d'un instrument pédagogique aussi puissant ; et nous devons, dans tous les cas, féliciter chaleureusement M. Édouard Lucas et ses éditeurs d'en avoir conçu l'idée et de l'avoir mis en pratique d'une façon aussi parfaite.

English translation of *La France*, 02/12/1889 (p. 3)

Under the title "Scientific Games to Serve History, Teaching, and the Practice of Calculation and Drawing," Messrs. Chambon and Baye have recently published a work that earned them a silver medal at the Universal Exhibition.

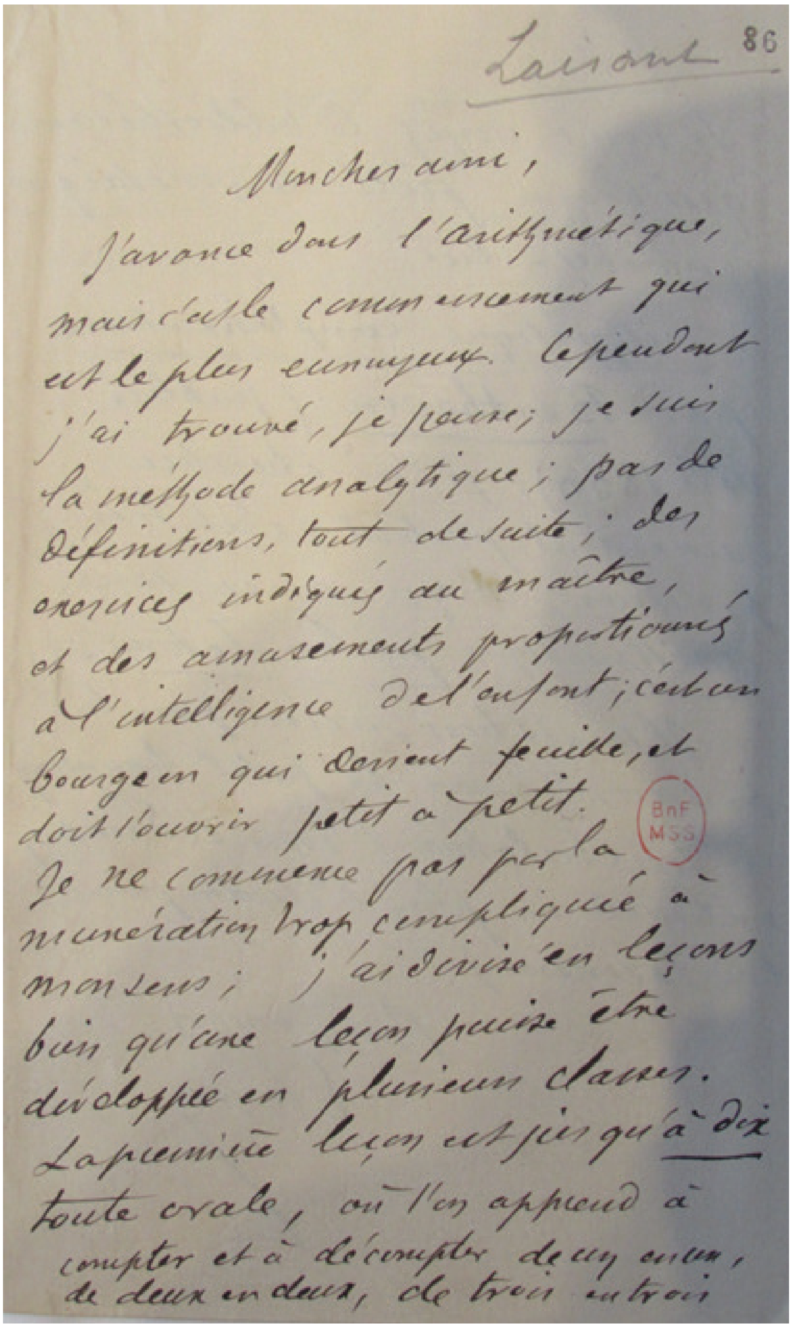
The author is Mr. Édouard Lucas, a professor of *Mathématiques spéciales* at *Lycée Saint-Louis* and one of our most distinguished scholars, especially in the branch of mathematics known as number theory.

Each game is enclosed in a very elegant box that also contains an explanatory booklet, part serious, part whimsical, always original. […]

It must be clearly understood that the frivolity of these games is only apparent. They sometimes touch upon the highest considerations of the science of combinations, and are, of a nature, in all cases, to make the study of calculation attractive, to develop taste, vision, and skill, especially concerning geometric drawing.

Educational establishments and families would be greatly mistaken to deprive themselves of such a powerful pedagogical tool. And we must, in all cases, warmly congratulate Mr. Édouard Lucas and his publishers for conceiving the idea and putting it into practice so perfectly.

Appendix 2 – Undated letter from Lucas to Laisant



Laisant 86

Mon cher ami,

J'avance dans l'arithmétique,
mais c'est le commencement qui
est le plus ennuyeux. Cependant
j'ai trouvé, je pense ; je suis
la méthode analytique ; pas de
définitions, tout de suite ; des
exercices indiqués au maître,
et des amusements proportionnés
à l'intelligence de l'enfant ; c'est un
bourgeon qui devient feuille, et
doit s'ouvrir petit à petit.
Je ne commence pas par la
numération trop compliquée à
mon sens ; j'ai divisé en leçons
bien qu'une leçon puisse être
développée en plusieurs classes.
La première leçon est jusqu'à dix
toute orale, où l'on apprend à
compter et à décompter de un en un,
de deux en deux, de trois en trois

Source: BnF NAF (Nouvelle acquisition française) n°28336 (fonds Laisant), fol. 86, recto

French Edition

Mon cher ami,

J'avance dans l'arithmétique [en référence à un ouvrage que Lucas est en train de rédiger] *mais c'est le commencement qui est le plus ennuyeux. Cependant j'ai trouvé, je pense ; je suis la méthode analytique ; pas de définitions, tout de suite ; des exercices indiqués au maître et des amusements proportionnés à l'intelligence de l'enfant ; c'est un bourgeon qui devient feuille, et doit s'ouvrir petit à petit.*

Je ne commence pas par la numération trop compliqué à mon sens ; j'ai divisé en leçons bien qu'une leçon puisse être développée en plusieurs classes.

*La première leçon est jusqu'*à dix *toute orale, où l'on apprend à compter et à décompter de un en un, de deux en deux, de trois en trois (fol. 86, verso) sur les doigts, ou avec des jetons. [...]*

English translation

My dear friend,

I am progressing in the Arithmetic [in reference to a book Lucas is currently writing], but it is the beginning that is the most tedious. However, I believe I have found; I am following the analytical method; no definitions right away; exercises indicated to the teacher and amusements proportionate to the child's intelligence; it's a bud that becomes a leaf and must open gradually. I do not start with numeration, which is too complicated in my opinion; I have divided it into lessons, although one lesson can be developed over several classes.

The first lesson is up to ten, all oral, where one learns to count and subtract one by one, two by two, three by three (fol. 86, verso) on fingers, or with tokens.

Histórico

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Como citar – APA

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