

Problems solving in the integral calculus and the determination of the area under the curve

Resolução de Problemas em Cálculo Integral
e Determinação da Área Sob a Curva

Resolución de Problemas en Cálculo Integral
y Determinación del Área Bajo la Curva

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ABSTRACT

In Integral Calculus the classic problem is the determination of the area under the curve, when said region is not expressible in terms of elementary figures. This translates into a multiplicity of problems and exercises that are presented to students in a Calculus course. This article presents a useful problem for Mathematics Education, derived from a generalized integral operator, for this we define what we understand by an integrable function in this generalized sense, and the geometric interpretation of a generalized definite integral is presented. The interesting thing about this generalization is that said geometric interpretation is similar to the geometric interpretation of the classical Riemann integral, but not in the xy plane, but in the Ty plane, where T is the kernel of the generalized integral.

Keywords: Problems solving; Integral calculus; Area under the curve.

RESUMO

o Cálculo Integral, o problema clássico é a determinação da área sob a curva, quando essa região não pode ser expressa em termos de figuras elementares. Isso se traduz em uma multiplicidade de problemas e exercícios apresentados aos alunos em um curso de Cálculo. Este artigo apresenta um problema útil para a Educação Matemática, derivado de um operador integral generalizado. Para isso, definimos o que entendemos por uma função integrável neste sentido generalizado, e a interpretação geométrica de uma integral definida generalizada é apresentada. O interessante dessa generalização é que a interpretação geométrica é semelhante à interpretação geométrica da integral clássica de Riemann, mas não no plano xy , e sim no plano Ty , onde T é o núcleo da integral generalizada.

Palavras-chave: Resolução de problemas; Cálculo integral; Área sob a curva.

RESUMEN

En Cálculo Integral, el problema clásico es la determinación del área bajo la curva, cuando dicha región no puede expresarse en términos de figuras elementales. Esto se traduce en una multiplicidad de problemas y ejercicios que se presentan a los estudiantes en un curso de Cálculo. Este artículo presenta un problema útil para la Educación Matemática, derivado de un operador integral generalizado. Para esto, definimos lo que entendemos por una función integrable en este sentido generalizado, y se presenta la interpretación geométrica de una integral definida generalizada. Lo interesante de esta generalización es que dicha interpretación geométrica es similar a la interpretación geométrica de la integral clásica de Riemann, pero no en el plano xy , sino en el plano Ty , donde T es el núcleo de la integral generalizada.

Palabras clave: Resolución de problemas; Cálculo integral; Área bajo la curva.

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INTRODUCTION

Problem Solving is probably one of the most developed trends in the last forty years and this development has been carried out in multiple directions: history and philosophy of Mathematics, constructivism, social constructivism, ethnomathematics, ... among others (readers interested in these various addresses can consult [41]).

Many researchers and potential employers have pointed out that school does not provide adequate attention to the understanding and skills that are necessary beyond school. In particular, the latter have pointed out that they need graduate students to be able to interpret and work with complex systems, to communicate and function with various specialists and to be able to continually adapt to new technologies [24].

One of the most fruitful mathematical branches for Problem Solving is Infinitesimal Calculus (Differential and Integral, in particular). The current division, which includes differential equations in addition to the previous ones, for example, has been a product of the historical development of Mathematics, where internal and external factors have led to the panorama we know today (see [17]). The differential and integral calculus seen from the geometric intuition can be observed in the concepts of Antiphon and Brison in their attempt to square the circle in order to determine its area and figures delimited by curves (area under the curve); this being the starting point for various mathematicians who formalized and developed the method such as Fermat, Newton, Leibniz, Cauchy, until reaching Weierstrass who rewrites the concepts and uses the known connotation found in current texts.

In particular, this development in the 18th century meant that mathematicians began to handle the current infinity with ease, and the curious thing is that this growth in the concepts of limit and convergence rests ultimately on the formalization of real numbers that are not he had even Peano and Cantor (see [36]). Interested readers can consult ([3, 5]) where a broad overview of the treatment of infinity in both mathematical research and Mathematics Education is shown. In fact, the logical and philosophical foundation of the differential and integral calculus was objectively impossible on the basis of the concepts on which they appeared and that is why the efforts of Newton, Leibniz, Lagrange and others, until the very beginning of the 19th century, ended in the failure. Let's point out the main shortcomings, apart from the lack of the concept of numerical domain and real numbers in particular that we already pointed out (see [21, 28, 39]):

Incorrect understanding of the concept of differential: In Leibniz, L'Hospital, Euler and other mathematicians of the 18th century, the concept of differential was confused in increment. A sufficiently correct approximation of the concept of differential was given only by Lagrange (1765).

Insufficient understanding of the concept of function: In fact, until the end of the 19th century, mathematicians, starting from mechanical and geometric intuition, understood by foundation only the analytical functions represented by a certain formula (in some cases infinite, as in Fourier's considerations linked to his theory of heat). Only with the appearance of discontinuous functions in practical problems did mathematicians pay attention to the logical formation of the concept of function (see last section of this lecture for more details).

Absence of a clear concept of limit: Newton's followers: Maclaurin, Taylor, Wallis and others, had a long discussion about whether or not the variable reaches the limit. This problem was not easy, precisely, because there was no precise definition of a limit and it was only determined by mechanical and geometric reasoning. This insufficiency remained until Cauchy (1823).

The concept of functional continuity was intuitive: This is explained because the mathematicians of the 18th century considered all functions continuous and therefore did not have the need to specify this concept. Only at the beginning of the 19th century did people begin to think about this problem (other details can be found in the last section of this lecture).

Fuzzy concept of definite integral: Related above all to the absence of an existence theorem. For example, it was considered that the Newton-Leibniz formula had a universal meaning, that is, it was valid for all functions and under all conditions. The efforts at the precision of the concept made by Lacroix, Poisson and Cauchy brought to the fore the concept of limit and continuity. But the problem of the definite integral only found a complete answer until the end of the 19th century in the works of Lebesgue.

The Integral Calculus is a mathematical area with so many ramifications and applications, that the sole intention of enumerating them makes the task practically impossible.

In this work we present a mathematical theoretical problem that has its impact on Mathematics Education, linked to the Resolution of Problems of the Integral Calculus. To do this, after placing the mathematical problem in its historical context, we point out some relevant contact points for Mathematics Education.

AN INTERLUDE ON PROBLEMS AND PROBLEM SOLVING

What is a problem? It is a good question, very difficult to answer, that is why we prefer to characterize what is a problem for us.

We can characterize a problem as a situation in which:

- There is a person interested in solving it (the solver).
- There is an initial situation (the current one) and a final one (where we want to get to when the problem is solved).
- The path to go from the initial situation to the final one is unknown.

What is a mathematical problem? Let's ask Chatgpt about it and see what he answers.

A mathematical problem is a question or situation that is addressed using mathematical principles and methods to find a solution or better understand the phenomenon in question. These problems can arise in a variety of areas, from physics and engineering to economics and biology, and generally involve the formulation of a clear question that requires mathematical reasoning to solve.

Mathematical problems can vary widely in complexity and focus. Some common examples include:

- Optimization problems: Determine the best possible solution under certain constraints, such as maximizing profits or minimizing costs.\
- Differential equation problems: Find functions that satisfy an equation involving derivatives.
- Geometry problems: Solve problems related to geometric shapes and figures, such as calculating areas, volumes or distances.\
- Probability and statistics problems: Analyze data and predict results based on probabilistic models.
- Number theory problems: Investigate properties of integers and numerical patterns.

Mathematical problems often require specific skills such as logical reasoning, the ability to formulate abstract models, the use of advanced mathematical techniques, and sometimes creativity to find innovative solutions. Solving mathematical problems not only expands our theoretical understanding, but also has significant practical applications in science, technology and engineering.

Specifically, as it could not be otherwise, Chatgpt answers what we already knew: a mathematical problem must be formulated and solved in the corresponding mathematical categories.

In some previous works we have referred to the role of mathematical problems in the development of Mathematics and its impact on Mathematics Education (see [10, 29, 30, 32, 33, 34]).

We know that the area problem is to definite integrals, what the tangent and rate of change problem is to derivatives.

Accumulation (or net change) problems are problems in which the rate of change of a quantity is given and we are asked to calculate the value of the quantity accumulated over time. These problems are solved using definite integrals. Derivatives are useful when we are given a quantity and asked about its rate, while integrals are useful when we are given a rate and asked about the quantity.

The applied problems are common in both differential and integral calculus. When we are presented with a problem, we must decide whether the solution involves derivatives or integrals. Of course, making the wrong decision will result in the wrong answer.

Problems and their resolution, linked to the defined integrations, have been studied in various works, from different points of view (see [1, 2, 6, 22]). Here we will approach it from the point of view of Mathematics, which distinguishes us from previous research.

A NEW INTEGRAL OPERATOR

We will assume that the fundamentals of the Riemann Integral are known by the attendees. In particular we will deal with real integral operators defined on $\mathbb{R}$.

One of the first operators that can be called fractional is that of Riemann-Liouville fractional derivatives of order $\alpha \in \mathbb{C}$, $Re(\alpha) \geq 0$, defined by (see [13]).

Definition 1.

Let $f \in L^1((a, b); \mathbb{R})$, $(a, b) \in \mathbb{R}^2$, $a < b$. The right and left side Riemann-Liouville fractional integrals of order $\alpha > 0$ are defined by

$${}^{RL}J_{a+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) ds, t > a \quad (1)$$

and

$${}^{RL}J_{b-}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (s-t)^{\alpha-1} f(s) ds, t < b \quad (2)$$

and their corresponding differential operators are given by

$$D_{a+}^{\alpha} f(t) = \frac{d}{dt} ({}^{RL}J_{a+}^{1-\alpha} f(t)) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t \frac{f(s)}{(t-s)^{\alpha}} ds$$

$$D_{b-}^{\alpha} f(t) = -\frac{d}{dt} ({}^{RL}J_{b-}^{1-\alpha} f(t)) = -\frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_t^b \frac{f(s)}{(s-t)^{\alpha}} ds$$

Other definitions of fractional operators are as follows.

Definition 2.

Let $f \in L^1((a, b); \mathbb{R})$, $(a, b) \in \mathbb{R}^2$, $a < b$. The right and left side Hadamard fractional integrals of order α with $Re(\alpha) > 0$ are defined by

$$H_{a+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{f(s)}{s} ds, \quad a < t < b \quad (3)$$

and

$$H_{b-}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b \left(\log \frac{s}{t}\right)^{\alpha-1} \frac{f(s)}{s} ds, \quad a < t < b \quad (4)$$

Hadamard differential operators are given by the following expressions.

$$({}^H D_{a+}^{\alpha} f)(t) = t \frac{d}{dt} (H_{a+}^{\alpha} f(t)) = \frac{-\Gamma(\alpha+1)}{B(\alpha, 1-\alpha)} \int_a^t \left(\log \frac{t}{s}\right)^{-\alpha-1} \frac{f(s)}{s} ds, \quad a < t < b$$

$$({}^H D_{b-}^{\alpha} f)(t) = -t \frac{d}{dt} (H_{b-}^{\alpha} f(t)) = -\frac{\Gamma(\alpha+1)}{B(\alpha, 1-\alpha)} \int_t^b \left(\log \frac{s}{t}\right)^{-\alpha-1} \frac{f(s)}{s} ds, \quad a < t < b$$

In [18], the author introduced new fractional integral operators, called the Katugampola fractional integrals, in the following way:

Definition 3.

Let $0 < a < b < +\infty$, $f : [a, b] \rightarrow \mathbb{R}$ is an integrable function, and $\alpha \in (0, 1)$ and $\rho > 0$ two fixed real numbers. The right and left side Katugampola fractional integrals of order α are defined by

$$K_{a+}^{\alpha, \rho} f(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t \frac{s^{\rho-1}}{(t^{\rho} - s^{\rho})^{1-\alpha}} f(s) ds, \quad a < t \quad (5)$$

and

$$K_{b-}^{\alpha, \rho} f(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_t^b \frac{t^{\rho-1}}{(s^{\rho} - t^{\rho})^{1-\alpha}} f(s) ds, \quad t < b \quad (6)$$

In [19], it appeared a generalization to the Riemann-Liouville and Hadamard fractional derivatives, called the Katugampola fractional derivatives:

$$({}^D_{a+}^{\alpha} f)(t) = \frac{\rho^{\alpha}}{\Gamma(1-\alpha)} t^{1-\rho} \frac{d}{dt} \int_a^t \frac{s^{\rho-1}}{(t^{\rho} - s^{\rho})^{\alpha}} f(s) ds, \quad a < t$$

$$({}^D_{b-}^{\alpha} f)(t) = \frac{-\rho^{\alpha}}{\Gamma(1-\alpha)} t^{1-\rho} \frac{d}{dt} \int_t^b \frac{s^{\rho-1}}{(s^{\rho} - t^{\rho})^{\alpha}} f(s) ds, \quad t < b$$

The relation between these two fractional operators is the following:

$$({}^D_{a+}^{\alpha, \rho} f)(t) = t^{1-\rho} \frac{d}{dt} K_{a+}^{1-\alpha, \rho} f(t), \quad ({}^D_{b-}^{\alpha, \rho} f)(t) = -t^{1-\rho} \frac{d}{dt} K_{b-}^{1-\alpha, \rho} f(t).$$

Definition 4.

Let I be an interval $I \subseteq \mathbb{R}$, $a, t \in I$ and, $\alpha \in \mathbb{R}$. The generalized integral operator $J_{T, a+}^{\alpha}$, right and $J_{T, b-}^{\alpha}$, left, is defined for every locally integrable function f on I as

$$J_{T, a+}^{\alpha}(f)(t) = \int_a^t \frac{f(s)}{T(t-s, \alpha)} ds, \quad t > a \quad (7)$$

and

$$J_{T, b-}^{\alpha}(f)(t) = \int_t^b \frac{f(s)}{T(s-t, \alpha)} ds, \quad b > t \quad (8)$$

Definition 5.

The lateral derivative operators (right and left) in the case of our generalized integral, are

$$(N_{T,a+}^{\alpha} f)(t) = N_T^{\alpha} [J_{T,a+}^{\alpha}(f)(t)] = \frac{d}{dx} [J_{T,a+}^{\alpha}(f)(t)] F(x, \alpha) \quad (9)$$

similarly to the left

Where N_T^{α} is the Generalized Derivative defined in [35] (see also [43])

Definition 6.

Given a function $f : [0, +\infty) \rightarrow \mathbb{R}$. Then the N-derivative of f of order α is defined by

$$N_T^{\alpha} f(t) = \lim_{\varepsilon \rightarrow 0} \frac{f(t + \varepsilon T(t, \alpha)) - f(t)}{\varepsilon} \quad (10)$$

for all $t > 0, \alpha \in (0, 1)$ being $T(\alpha, t)$ is some function. Here we will use some cases of T defined in function of $E_{a,b}(\cdot)$ the classic definition of

Mittag-Leffler function with $Re(a), Re(b) > 0$. Also we consider $E_{a,b}(t^{-\alpha})_k$ is the k-nth term of $E_{a,b}(\cdot)$.

If f is α -differentiable in some $(0, \alpha)$, and $\lim_{t \rightarrow 0^+} N_T^{(\alpha)} f(t)$ exists, then define $N_T^{(\alpha)} f(0) = \lim_{t \rightarrow 0^+} N_T^{(\alpha)} f(t)$, note that if f is differentiable, then $N_T^{(\alpha)} f(t) = T(t, \alpha) f'(t)$ where $f'(t)$ is the ordinary derivative.

The original function $E_{\alpha,1}(z) = E_{\alpha}(z)$ was defined and studied by Mittag-Leffler in the year 1903, that is, a uniparameter function (see [25, 26]). It is a

direct generalization of the exponential function. Wiman proposed and studied a generalization of the role of Mittag-Leffler, who we'll call it the Mittag-Leffler function with two parameters $E_{\alpha,\beta}(z)$ (see [42]), Agarwal in 1953 and Humbert and Agarwal in 1953, also made contributions to the final formalization of this function.

It is easy to see that the case of the J_T^{α} operator defined above contains, as particular cases, the integral operators obtained from conformable and non-conformable local derivatives. However, we will see that it goes much further by containing the cases listed at the beginning of the work. So, we have

1) If $F(t, \alpha) = t^{1-\alpha}$, $T(t, \alpha) = \Gamma(\alpha)F(t - s, \alpha)$, from (7) we have the right side Riemann-Liouville fractional integrals $(R_{a+}^{\alpha} f)(t)$, similarly from (8) we obtain the left derivative of Riemann-Liouville. Then its corresponding right differential operator is

$$({}^{RL}D_{a+}^{\alpha} f)(t) = \frac{d}{dt} (R_{a+}^{1-\alpha} f)(t),$$

analogously we obtain the left.

2) With $F(t, \alpha) = t^{1-\alpha}$, $T(t-s, \alpha) = \Gamma(\alpha)F(\ln t - \ln s, \alpha)t$, we obtain the right Hadamard integral from (7), the left Hadamard integral is obtained similarly from (8). The right derivative is

$$({}^H D_{a+}^{\alpha} f)(t) = t \frac{d}{dt} (H_{a+}^{1-\alpha} f)(t),$$

in a similar way we can obtain the left

3) The right Katugampola integral is obtained from (7) making

$$F(t, \alpha) = t^{1-\alpha}, \quad e(t) = t^{\rho}, \quad T(t, \alpha) = \frac{\Gamma(\alpha)}{F(\rho, \alpha)} \frac{F(e(t) - e(s), \alpha)}{e'(s)},$$

analogously for the left fractional integral. In this case, the right derivative is

$$({}^K D_{a+}^{\alpha, \rho} f)(t) = t^{1-\rho} \frac{d}{dt} K_{a+}^{1-\alpha, \rho} f(t) = F(t, \rho) \frac{d}{dt} K_{a+}^{1-\alpha, \rho} f(t),$$

and we can obtain the left derivative in the same way.

4) The solution of equation $(-\Delta)^{-\frac{\alpha}{2}} \phi(u) = -f(u)$ called Riesz potential, is given by the expression $\phi = C_n^{\alpha} \int_{\mathbb{R}^n} \frac{f(v)}{|u-v|^{n-\alpha}} dv$, where C_n^{α} is a constant (see [9, 15, 27]). Obviously, this solution can be expressed in terms of the operator (7) very easily.

5) It is clear then, that from our definition, new extensions and generalizations of known integral operators can be defined. For example, in KMS presented the definition of fractional integral of f with respect to g of following way. Let $g : [a, b] \rightarrow \mathbb{R}$ be an increasing and positive monotone function on $(a, b]$ having a continuous derivative $g'(t)$ on (a, b) . The left-sided fractional integral of f with respect to the function g on $[a, b]$ of order $\alpha > 0$ is defined by

$$I_{g, a+}^{\alpha}(f)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{g'(s)f(s)}{[g(t) - g(s)]^{1-\alpha}} ds, \quad t > a, \quad (11)$$

similarly the right lateral derivative is defined as well

$$I_{g, b-}^{\alpha}(f)(t) = \frac{1}{\Gamma(\alpha)} \int_t^b \frac{g'(s)f(s)}{[g(s) - g(t)]^{1-\alpha}} ds, \quad t < b. \quad (12)$$

It will be very easy for the reader to build the kernel T in this case.

6) A k -analogue of above definition is defined in [23] (also see [11]), under the same assumptions on function g

$$I_{g, a+}^{\alpha, k}(f)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{g'(s)f(s)}{[g(t) - g(s)]^{1-\frac{\alpha}{k}}} ds, \quad t > a, \quad (13)$$

similarly the right lateral derivative is defined as well

$$I_{g, b-}^{\alpha, k}(f)(t) = \frac{1}{\Gamma(\alpha)} \int_t^b \frac{g'(s)f(s)}{[g(s) - g(t)]^{1-\frac{\alpha}{k}}} ds, \quad t < b \quad (14)$$

The corresponding differential operator is also very easy to obtain.

7) We can define the function space $L^p_\alpha[a, b]$ as the set of functions over $[a, b]$ such that $(J_{T, a+}^\alpha[f(t)]^p(b)) < +\infty$.

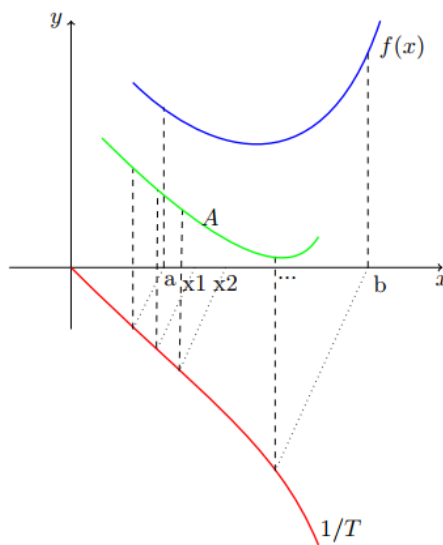
TOWARDS A GEOMETRIC INTERPRETATION OF THE GENERALIZED INTEGRAL

Definition 7.

Given a closed interval $I = [a, b]$, a partition of I is any finite strictly increasing sequence of points $P = x_0, x_1, \dots, x_{n-1}, x_n$ such that $a = x_0$ and $b = x_n$. The norm of the partition P is defined by

$$\|P\| = \max_{1 \leq j \leq n} (x_j - x_{j-1}) = \Delta x_j \quad (15)$$

If f is a function whose domain contains the closed interval I and f is bounded on the interval I , we know that f has both a least upper bound and a greatest lower bound on I as well as on each interval of any subdivision of I .



By the absolute continuity of $\frac{1}{T(t, \alpha)}$ the partition P on $[a, b]$ generates other partition P_T on $\left[\min_{t \in [a, b]} \frac{1}{T(t, \alpha)}, \max_{t \in [a, b]} \frac{1}{T(t, \alpha)}\right]$ with $x_0^T = \min_{t \in [a, b]} \frac{1}{T(t, \alpha)}$ and $x_n^T = \max_{t \in [a, b]} \frac{1}{T(t, \alpha)}$ and denoting by the norm of this new partition $\|P^T\| = \max_{1 \leq j \leq n} (x_j^T - x_{j-1}^T) = \Delta x_j^T$.

Let's define $S(f, T, \alpha) = \sum_{j=0}^n \frac{\max_j f(x)}{\Delta x_j^T}$, and $s(f, T, \alpha) = \sum_{j=0}^n \frac{\min_j f(x)}{\Delta x_j^T}$, thus we can state our criterion of integrability as follows.

Definition 8.

The function f defined and bounded on $[a, b]$ is said to be N -integrable (or integrable in a general sense) if for all $\varepsilon > 0$ there is a partition P_T such that

$$|S(f, T, a) - s(f, T, a)| < \varepsilon, \quad (16)$$

when

$$\lim_{j \rightarrow \infty} \Delta x_j^T = 0 \quad (17)$$

Remark 9.

A sufficient condition to guarantee the convergence of (16) is the Cauchy Mayorante Criterion, that is, that the inequality is satisfied if there is a numeric series a_j such that

$$|S(f, T, a) - s(f, T, a)| < a_j, \quad (18)$$

and $\sum_{j=0}^{\infty} a_j < +\infty$. This is possible, for example, if T besides being uniformly continuous is strictly decreasing.

The geometric interpretation of the integral $J_{T,a}^{\alpha}(f)(b)$ is as before, the area under the curve, but not with respect to $[a, b]$, if not, with respect to the new interval generated by $\frac{1}{T}$, the area is represented by the letter A in the previous graph. What we want to highlight is that, even when we work with the new integral operators, the geometric interpretation remains the same!

CONCLUSIONS

We know that problem solving is very important in the curriculum at any level of education, particularly at the higher level. Problem solving is a process of educational objectives that can be seen from the curricular aspect ([40]). Problem solving is a process of applying previously obtained information to new information, for unknown situations ([20]). The application of problem solving requires skills such as critical thinking, so that when students master a problem, they can make plans to solve them and be able to find the correct answer when faced with an unknown situation ([7]).

Differential and integral calculus is present in the curricula of university degrees in exact and natural sciences, engineering, technology and related professorships. Reflections and discussions on what, how and what to teach in Calculus have stimulated his study from different perspectives. For instance, in the study of mathematical thought processes, visualization has been a growing area of development in recent decades. Much of this research points to visual reasoning as a powerful and alternative resource for students when doing math (see [4, 16, 37, 44]). We consider that in this last direction this work can contribute resources, using a table similar to y related to the fundamentals of the exhaustion method), using different kernels, remember that if $T \equiv 1$ we have the classic Riemann Integral.

α	kernel	Area
-	$T \equiv 1$	0,3333..
$\frac{1}{2}$	$x^{1-\alpha}$	0,4
$\frac{1}{3}$	$x^{-\alpha}$	0,3
$\frac{2}{5}$	$e^{(1-\alpha)x}$	0,2140304...
$\frac{9}{10}$	$e^{x^{-\alpha}}$	0,089093...

Besides it is clear that, under the Definitions 4 and 5 many of the results reported in the literature, for the derivatives and integrals presented above as particular cases, can be extended without much difficulty. On the other hand, we have shown that any integral operator, included in Definition 3, can be formalized with a scheme similar to that of the Riemann Integral, that is, by means of higher and lower sums.

For all of the above, we want to emphasize that problem solving is essential in learning integral calculus because it allows students to apply theoretical concepts in practical and real situations. When faced with problems, students develop critical thinking skills, logical reasoning, and the ability to find creative solutions. In addition, problem solving helps to consolidate the knowledge acquired, identify possible errors and improve the understanding of mathematical concepts. In summary, constant practice of solving problems in integral calculus is essential for the development of strong mathematical skills and preparation to meet academic and professional challenges.

In particular, we believe that the problem presented in this manuscript would help consolidate both theoretical knowledge and the geometric interpretation of the definite integral, seen as the area under the curve.

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